On the Regularity of Maximal Functions

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Abstract

In this diploma thesis we discuss about the Hardy-Littlewood maximal operator and some results on the regularity properties. We distinguish the continuous and the discrete formulations. We explain theorems concerning the maximal function of a Sobolev function and the maximal function of a function of bounded variation. The final chapter is devoted to present some convolution type maximal operators and their regularity properties. The philosophy of this work is to determine if these operators preserve the differentiability and the variation of the functions and if we can give estimates to control it.

Chapter 1

The Hardy-Littlewood maximal function

The purpose of this report is to present some studies on the regularity properties of the Hardy-Littlewood maximal operator. Maximal operators are central objects in harmonic analysis with applications to pointwise convergence of Fourier series, ergodic theorems and pointwise convergence of solutions of partial differential equations [3, 9]. The Hardy-Littlewood maximal operator is well known for its application to the Lebesgue Differentiation Theorem. Other applications to PDEs suggest that it is important to study how the maximal operator interacts with the regularity of the functions, see references in [7]. This means to study if the maximal operator keeps, destroys or improves the weak differentiability of the initial function.

In this chapter, we will present some definitions and basic properties needed to understand the subsequent chapters. This part of the report is based on classical literature [5, 10, 11].

1.1 Definition and elementary properties

We will start with the following definition of our central object of discussion. Let $f \in L_{loc}(\mathbb{R}^d)$. The **centered** maximal function of f is

$$Mf(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|dy. \tag{1.1}$$

The **non-centered** maximal function of f is

$$\widetilde{M}f(x) = \sup_{x \in B} \frac{1}{m(B)} \int_{B} |f(y)|dy. \tag{1.2}$$

In these definitions, $B(x,r)$ represents the open ball centered at x and radius r, $m(B(x,r))$ denotes its Lebesgue measure. In (1.2) B is any open ball and we do not specify the center, we only require that the ball contains x.

We can consider the **maximal operator** (non-centered) as $f \mapsto Mf$ ($f \mapsto \widetilde{M}f$). The definition of this object has a very nice motivation, the Lebesgue Differentiation Theorem; when we consider the averages

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy$$

and ask whether the limit $A_r f \to f$ holds almost everywhere as $r \to 0$. Note that $Mf(x) = \sup_{r>0} A_r|f|(x)$.

We prove the following fundamental theorem, we will use it several times across our discussions. This is a classical fact about the Hardy-Littlewood maximal operator, see [10, 11].

Theorem 1.1 (Weak and strong type inequalities)**.** *Let f be a function defined in $\mathbb{R}^d$.*

(i) If $f \in L^p(\mathbb{R}^d)$, $1 \leq p \leq \infty$, then the function $\widetilde{M}f$ is finite almost everywhere.

(ii) If $f \in L^1(\mathbb{R}^d)$, then for every $\alpha > 0$,

$$m\{x : \widetilde{M}f(x) > \alpha\} \leq \frac{A}{\alpha}||f||_1 \tag{1.3}$$

where A depends only on the dimension d.

(iii) If $f \in L^p(\mathbb{R}^d)$, $1 < p \leq \infty$, then the function $\widetilde{M}f \in L^p(\mathbb{R}^d)$ and

$$||\widetilde{M}f||_p \leq A_p||f||_p \tag{1.4}$$

where A_p depends only on the value of p and the dimension d.

It is important to comment that Theorem 1.1 holds for the centered maximal function as well, since the non-centered maximal function is computed by taking more averages than the centered, consequently we have the inequality $Mf \leq \widetilde{M}f$ everywhere.

The conclusion in part (iii) is false for $p = 1$ unless $f = 0$ almost everywhere. To see this, let R be a real number bigger than 1, and notice that for $x \in \mathbb{R}^d$ such that $|x| \geq R$, we have

$$\begin{aligned} Mf(x) &\geq \frac{1}{m(B(x, R+1))} \int_{B(x,R+1)} |f(y)|dy \\ &\geq \frac{\int_{B(0,1)|f(y)|dy}}{m(B(0,1))} \frac{1}{(R+1)^d} \\ &\geq C \frac{1}{(|x|+1)^d}. \end{aligned}$$

If we assume without loss of generality that $\int_{B(0,1)} |f(y)|dy \neq 0$, the constant C above is positive. It is true that Mf is not L^1 since the lower bound $\frac{1}{(|x|+1)^d}$ is not integrable at infinity.

Before going to the proof of Theorem 1.1, we will prove the following useful fact:

Lemma 1.2 (Vitali covering)**.** *Let E a measurable subset of $\mathbb{R}^d$ that is the union of a finite number of balls $\{B_j\}$, then one can select a disjoint subcollection $B_1, \dots, B_m$ such that*

$$m(E) \leq A \sum_{k=1}^{m} m(B_k) \tag{1.5}$$

with $A > 0$. The choice $A = 3^d$ works.

Proof. We extract the desired balls using the following procedure, first let B_1 one of the balls in $\{B_j\}$ with the largest radius. If there are disjoint balls to B_1, select B_2 with the highest radius. Now, if there are balls in $\{B_j\}$ disjoint to B_1 and B_2, select B_3 as the one with largest radius. These iterations finish with a subcollection $B_1, \dots, B_k, \dots, B_m$, ordered by decreasing radius. To distinguish this subcollection to the original $\{B_j\}$, we use the subscript k instead of j.

Define B^* as a ball with the same center than B but 3 times the radius. We claim that $\cup B_k^* \supseteq E$. To prove this, it is enough to show that $\cup B_k^* \supseteq B_j$ for every ball in the collection $\{B_j\}$. Let B_j one of the balls in $\{B_j\}$ which is not one of $B_1, \dots, B_k, \dots, B_m$ otherwise the result is trivial. This implies that B_j intersects one of the balls $B_1, \dots, B_m$ say B_{j_0}, using triangle inequality $B_j \subseteq B_{j_0}^*$. We conclude that

$$m(E) \leq \sum_{k=1}^{m} m(B_k^*) = 3^d \sum_{k=1}^{m} m(B_k)$$

$\square$

Proof of Theorem 1.1. Part (i) follows immediately from part (ii) and (iii).

We prove part (ii). Notice that $\widetilde{M}f$ is lower semi-continuous; let $x_0 \in \mathbb{R}^d$ and suppose $\widetilde{M}f(x_0) < \infty$, given $\epsilon > 0$, there exists an open ball B containing x_0 such that $1/m(B)\int_B |f(y)|dy > \widetilde{M}f(x_0) - \epsilon$. Hence for every $x \in B$, we have that $\widetilde{M}f(x) > \widetilde{M}f(x_0) - \epsilon$. If $\widetilde{M}f(x_0) = \infty$, let B_N a ball containing x_0 such that $1/m(B_N)\int_{B_N} |f(y)|dy \geq N$. An arbitrary sequence $\{x_n\}$ that converges to x_0 satisfies that $\widetilde{M}f(x_n) \geq N$ for all n sufficiently large, so $\widetilde{M}f(x_n) \to \infty$. Now, let $\alpha > 0$ and define the open set $E_\alpha = \{x \in \mathbb{R}^d; \widetilde{M}f(x) > \alpha\}$. Fix a compact set $E \subset E_\alpha$, by lemma 1.2 we can find a finite sequence of disjoint open balls $B_1, \ldots, B_m$ contained in E_α and satisfying (1.5). Since these balls are disjoint, we have that

$$c^{-1}m(E) \leq \sum_{k=1}^{m} m(B_k) \leq \frac{1}{\alpha}\int_{\cup_{k=1}^m B_k} |f(y)|dy \leq \frac{1}{\alpha}||f||_1 \tag{1.6}$$

The desired result follows by taking the supremum on left side of (1.6) and the inner regularity of the Lebesgue measure.

The proof of part (iii) is as follows, notice that the case $p = \infty$ is trivial with $A_\infty = 1$. The case $1 < p < \infty$ follows from the part (ii), the case $p = \infty$ and the Marcinkiewicz Interpolation Theorem 1.3, since the maximal function is a sublinear operator as we will see in the next section. □

1.2 Marcinkiewicz interpolation theorem

Now we turn our attention to an important tool of interpolation, this is what we need to complete the proof of the Theorem 1.1 part (iii). For the sake of our exposition we state the theorem first, but the new terminology introduced here will be explained in detail just before the proof. The proof is based on the one given in [5], we adjusted the statement to Lebesgue measure, but it still holds for more general measures.

Theorem 1.3 (Marcinkiewicz interpolation theorem)**.** *Let $p_0, p_1, q_0, q_1 \in [1, \infty]$ such that $p_0 \leq q_0$ and $p_1 \leq q_1$, and $q_0 \neq q_1$. For $0 < t < 1$, denote*

$$\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1} \quad \textit{and} \quad \frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}.$$

If T is a sublinear operator from $L^{p_0} + L^{p_1}$ to the space of measurable functions of $\mathbb{R}^d$ of weak types (p_0, q_0) and (p_1, q_1), then T is of strong type (p, q). More precisely, if there exist positive constants C_0 and C_1, such that $[Tf]_{q_i} \leq C_i||f||_{p_i}$ for $j = 0, 1$.

As we already said, we need several definitions and some propositions. If f is a measurable function on $\mathbb{R}^d$, the **distribution function** $\lambda_f : (0, \infty) \to [0, \infty]$ is defined as

$$\lambda_f(\alpha) = m(\{x : |f(x)| > \alpha\}).$$

The distribution function has some basic properties:

Proposition 1.4. *(i)* λ_f *is non-increasing and right continuous.*

(ii) If $|f| \leq |g|$*, then* $\lambda_f \leq \lambda_g$*.*

(iii) If $|f_n|$ *increases to* $|f|$*, then* λ_{f_n} *increases to* λ_f*.*

(iv) If $f = g + h$*, then* $\lambda_f(\alpha) \leq \lambda_g(\frac{\alpha}{2}) + \lambda_h(\frac{\alpha}{2})$*.*

Proof. Part (i) If $\alpha \leq \beta$, we have the inclusion $E_f(\beta) := \{x : |f(x)| > \beta\} \subseteq \{x : |f(x)| > \alpha\} = E_f(\alpha)$ therefore, by monotonicity of the Lebesgue measure $\lambda_f(\beta) \leq \lambda_f(\alpha)$. Since the union of nested sets $E_f(\alpha) = \cup_n E_f(\alpha + n^{-1})$ holds, the right continuity follows.
Part (ii) If $|f(x)| \leq |g(x)|\ \forall x \in \mathbb{R}^d$, then the inclusion $E_f(\alpha) \subset E_g(\alpha)$ holds and the inequality $\lambda_f(\alpha) \leq \lambda_g(\alpha)$ is true for any $\alpha > 0$.
Part (iii) If $|f_n|$ increases to $|f|$, then $\{E_{f_n}(\alpha)\}$ is an increasing nested sequence of sets and $\lambda_{f_n}(\alpha) \uparrow \lambda_f(\alpha)$.
Part (iv) If $|g(x)| + |h(x)| \geq |f(x)| > \alpha$ it follows that $|g(x)| > \frac{\alpha}{2}$ or $|h(x)| > \frac{\alpha}{2}$, and we obtain the inclusion $E_f(\alpha) \subseteq E_g(\frac{\alpha}{2}) \cup E_h(\frac{\alpha}{2})$ which implies the conclusion. $\square$

An important application of the distribution function is that we can rewrite the p-norms in terms of it. If $f \in L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$, by the Fubini's theorem

$$\begin{aligned} ||f||_p^p &= \int_{\mathbb{R}^d} \int_0^{|f(x)|} p\alpha^{p-1} d\alpha\, dx \\ &= \int_0^{\infty} \int_{|f(x)|>\alpha} p\alpha^{p-1} dx\, d\alpha = p \int_0^{\infty} \alpha^{p-1} \lambda_f(\alpha) d\alpha, \end{aligned} \tag{1.7}$$

and if $p = \infty$, then

$$||f||_\infty = \inf\{\alpha; \lambda_f(\alpha) = 0\}. \tag{1.8}$$

Now we give some definitions about weak $L^p(\mathbb{R}^d)$ spaces. If f is a measurable function on $\mathbb{R}^d$ and $1 \leq p < \infty$, we define

$$[f]_p = \left(\sup_{\alpha>0} \alpha^p \lambda_f(\alpha) \right)^{1/p},$$

and we define **weak** $L^p(\mathbb{R}^d)$ as the collection of measurable functions f for which $[f]_p < \infty$. We can say more about the weak $L^p(\mathbb{R}^d)$ spaces:

- The quantity $[\cdot]_p$ is not a norm
- The weak $L^p(\mathbb{R}^d)$ spaces are Banach spaces with a topology generated by $[\cdot]_p$.
- The inclusion $L^p(\mathbb{R}^d) \subset$ weak $L^p(\mathbb{R}^d)$ holds.
- The inequality $[f]_p \leq ||f||_p$ is always true.

Now we describe a way to decompose a measurable function f and prove some relation of this decomposition with the distribution λ_f: let $A > 0$, as before $E_f(A) = \{x : |f(x)| > A\}$, define

$$h_A = f\chi_{\mathbb{R}^d \setminus E_f(A)} + (\operatorname{sgn} f)A\chi_{E_f(A)}, \qquad g_A = f - h_A = (\operatorname{sgn} f)(|f| - A)\chi_{E_f(A)},$$

where χ_E is the characteristic or the indicator function of the set E and $\operatorname{sgn} f$ is the sign function of f.

Proposition 1.5. *If f is a measurable function in $\mathbb{R}^d$, then*

$$\lambda_{h_A}(\alpha) = \begin{cases} \lambda_f(\alpha) & \text{if } \alpha < A, \\ 0 & \text{if } \alpha \geq A. \end{cases} \qquad \lambda_{g_A}(\alpha) = \lambda_f(\alpha + A).$$

Proof. Since $|h_A| \leq A$ everywhere for $\alpha \geq A$, the set $E_{h_A}(\alpha)$ is empty, then $\lambda_{h_A}(\alpha) = 0$. If $\alpha < A$, the set $E_{h_A}(\alpha)$ is equal to the set $E_f(\alpha)$, therefore $\lambda_{h_A}(\alpha) = \lambda_f(\alpha)$. Let $\alpha > 0$, we have $E_f(\alpha + A) = \{x; |f(x)| > \alpha + A\} = \{x; (|f(x)| - A)\chi_{E_f(A)} > \alpha\} = E_{g_A}(\alpha)$, therefore $\lambda_{g_A}(\alpha) = \lambda_f(\alpha + A)$. □

Now we define three more concepts related to sublinear operators and its type. First of all, an operator T defined on a vector space $\mathcal{S}$ of measurable functions defined on $\mathbb{R}^d$ over measurable functions on $\mathbb{R}^d$ is said to be **sublinear** if for every $f, g \in \mathcal{S}$, $|T(f + g)| \leq |Tf| + |Tg|$ and for every positive constant c, the equality $|T(cf)| = c|Tf|$ holds. Secondly, a sublinear map T is of **strong type** (p, q) for $1 \leq p, q \leq \infty$, if T is a bounded operator that maps $L^p(\mathbb{R}^d)$ to $L^q(\mathbb{R}^d)$, in other words, it exists $C > 0$ such that $||Tf||_q \leq C||f||_p$ for all $f \in L^p(\mathbb{R}^d)$. Finally, we said that a sublinear operator T is of **weak type** (p, q) for $1 \leq p \leq \infty$ and $1 \leq q < \infty$, if T is a bounded operator that maps $L^p(\mathbb{R}^d)$ into weak $L^q(\mathbb{R}^d)$, in other words, it exists $C > 0$ such that $[Tf]_q \leq C||f||_p$ for all $f \in L^p$. A sublinear operator T is of weak type (p, ∞) if and only if it is of strong type (p, ∞).

Proof Theorem 1.3. We divide the proof in 3 cases.

First, suppose that $p_0 = p_1 =: p$, with no loss of generality, we can assume $q_0 < q_1$ (it implies that $q_0 < q < q_1$). The inequalities $[Tf]_{q_0} \leq C_0||f||_p$ and $[Tf]_{q_1} \leq C_1||f||_p$ imply

$$\lambda_{Tf}(\alpha) \leq (C_0||f||_p/\alpha)^{q_0} \qquad \lambda_{Tf}(\alpha) \leq (C_1||f||_p/\alpha)^{q_1} .$$

Let $0 < \sigma < \infty$, we have

$$\begin{aligned}
||Tf||_q^q &= q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha \\
&\leq q\int_0^\sigma \alpha^{q-1}\left(\frac{C_0||f||_p}{\alpha}\right)^{q_0} d\alpha + q\int_\sigma^\infty \alpha^{q-1}\left(\frac{C_1||f||_p}{\alpha}\right)^{q_1} d\alpha \qquad (1.9)\\
&= \frac{q(C_0||f||_p)^{q_0}\sigma^{q-q_0}}{|q-q_0|} + \frac{q(C_1||f||_p)^{q_1}\sigma^{q-q_1}}{|q-q_1|}.
\end{aligned}$$

For instance, if we choose $\sigma = \left(\frac{C_1^{q_1}}{C_0^{q_0}}\right)^{\frac{1}{q_1-q_0}} ||f||_p$, we find that

$$\begin{aligned}
&\sup\{||Tf||_q; ||f||_p = 1\} \\
&\leq B := \left(\frac{qC_0^{q_0}}{|q-q_0|}\left(\frac{C_1^{q_1}}{C_0^{q_0}}\right)^{\frac{q-q_0}{q_1-q_0}} + \frac{qC_1^{q_1}}{|q-q_1|}\left(\frac{C_1^{q_1}}{C_0^{q_0}}\right)^{\frac{q-q_1}{q_1-q_0}}\right)^{\frac{1}{q}}. \qquad (1.10)
\end{aligned}$$

Finally, notice that for a real positive number $c > 0$, the operator satisfies the equality $|T(cf)| = c|Tf|$, we conclude that $||Tf||_q \leq B||f||_p$. For the remaining cases, the idea is to use the decomposition of f that we introduced before the Proposition 1.5, with a clever choice of A.

Second, suppose without loss of generality $p_0 < p_1$ and $q_0 < \infty$, $q_1 < \infty$, the hypothesis implies that $p_0 < p_1 < \infty$. Using the formula given by (1.7) and Proposition 1.5, we have that

$$\begin{aligned}
\int |h_A|^{p_1}dx &= p_1\int_0^\infty \beta^{p_1-1}\lambda_{h_A}(\beta)d\beta = p_1\int_0^A \beta^{p_1-1}\lambda_f(\beta)d\beta, \\
\int |g_A|^{p_0}dx &= p_0\int_0^\infty \beta^{p_0-1}\lambda_{g_A}(\beta)d\beta = p_0\int_0^\infty \beta^{p_0-1}\lambda_f(\beta+A)d\beta \qquad (1.11)\\
&= p_0\int_A^\infty (\beta-A)^{p_0-1}\lambda_f(\beta)d\beta \leq p_0\int_A^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta.
\end{aligned}$$

Using the sublinearity of the operator T, parts (ii) and (iv) of Proposition 1.4, we can estimate the following:

$$\begin{aligned}\int |Tf|^q dx &= q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(\alpha)d\alpha = 2^q q\int_0^\infty \alpha^{q-1}\lambda_{Tf}(2\alpha)d\alpha \\ &\le 2^q q\int_0^\infty \alpha^{q-1}\left(\lambda_{Tg_A}(\alpha)+\lambda_{Th_A}(\alpha)\right)d\alpha.\end{aligned} \tag{1.12}$$

By definition $p = p_0p_1/(p_1(1-t)+p_0t)$ and $q = q_0q_1/(q_1(1-t)+q_0t)$. Let σ be

$$\begin{aligned}\sigma := \frac{p_0(q_0-q)}{q_0(p_0-p)} &= \frac{1-\frac{q}{q_0}}{1-\frac{p}{p_0}} = \frac{1-\frac{q_1}{q_1(1-t)+q_0t}}{1-\frac{p_1}{p_1(1-t)+p_0t}} = \frac{p^{-1}(q_1^{-1}-q_0^{-1})}{q^{-1}(p_1^{-1}-p_0^{-1})} \\ = \frac{1-\frac{q_0}{q_1(1-t)+q_0t}}{1-\frac{p_0}{p_1(1-t)+p_0t}} &= \frac{1-\frac{q}{q_1}}{1-\frac{p}{p_1}} = \frac{p_1(q_1-q)}{q_1(p_1-p)}.\end{aligned} \tag{1.13}$$

Since (1.11) and (1.12) hold for every choice of A, let A be equal to α^σ. Therefore, we can estimate that the q-norm of Tf as follows

$$\begin{aligned}||Tf||_q^q &\le 2^q q\int_0^\infty \alpha^{q-1}\left[([Tg_A]_{q_0}/\alpha)^{q_0} + ([Th_A]_{q_1}/\alpha)^{q_1}\right]d\alpha \\ &\le 2^q q\int_0^\infty \alpha^{q-1}\left[(C_0||g_A||_{p_0}/\alpha)^{q_0} + (C_1||h_A||_{p_1}/\alpha)^{q_1}\right]d\alpha \\ &\le 2^q qC_0^{q_0}\int_0^\infty \alpha^{q-q_0-1}\left(p_0\int_A^\infty \beta^{p_0-1}\lambda_f(\beta)d\beta\right)^{q_0/p_0} d\alpha \\ &\quad + 2^q qC_1^{q_1}\int_0^\infty \alpha^{q-q_1-1}\left(p_1\int_0^A \beta^{p_1-1}\lambda_f(\beta)d\beta\right)^{q_1/p_1} d\alpha.\end{aligned} \tag{1.14}$$

If we denote, χ_0 and χ_1 the characteristic functions of the sets $\{(\alpha,\beta);\beta>\alpha^\sigma\}$ and $\{(\alpha,\beta);\beta<\alpha^\sigma\}$ and $\phi_j(\alpha,\beta) = \chi_j(\alpha,\beta)\alpha^{(q-q_j-1)p_j/q_j}\beta^{p_j-1}\lambda_f(\beta)$, we can conclude from (1.14) that

$$||Tf||_q^q \le \sum_{j=0}^1 2^q qC_j^{q_j}p_j^{q_j/p_j}\int_0^\infty\left(\int_0^\infty \phi_j(\alpha,\beta)d\beta\right)^{q_j/p_j} d\alpha. \tag{1.15}$$

By the assumptions of this case, $1\le q_j/p_j<\infty$, for $j=0,1$. This allows us to use the Minkowski's inequality and conclude that

$$\begin{aligned}\int_0^\infty \left(\int_0^\infty \phi_j(\alpha,\beta)\,d\beta\right)^{\frac{q_j}{p_j}} d\alpha \\ \le \left(\int_0^\infty \left(\int_0^\infty \phi_j(\alpha,\beta)^{\frac{q_j}{p_j}}\,d\alpha\right)^{\frac{p_j}{q_j}} d\beta\right)^{\frac{q_j}{p_j}}.\end{aligned} \tag{1.16}$$

If $q_0 < q_1$, we have that $\frac{1}{q_1} < \frac{1}{q} < \frac{1}{q_0}$ (and $q_0 < q < q_1$), also we assumed $p_0 < p_1$, so $p_0 < p < p_1$. Consequently, σ is positive and we have the identities

$$\begin{aligned}\int_0^\infty \left(\int_0^\infty \phi_0(\alpha,\beta)^{\frac{q_0}{p_0}}\,d\alpha\right)^{\frac{p_0}{q_0}} d\beta &= \int_0^\infty \left(\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1}(\beta^{p_0-1}\lambda_f(\beta))^{\frac{q_0}{p_0}}\,d\alpha\right)^{\frac{p_0}{q_0}} d\beta \\ &= \int_0^\infty \left(\int_0^{\beta^{1/\sigma}} \alpha^{q-q_0-1} d\alpha\right)^{\frac{p_0}{q_0}} \beta^{p_0-1}\lambda_f(\beta)d\beta \\ &= \int_0^\infty \left(\frac{\beta^{(q-q_0)/\sigma}}{q-q_0}\right)^{\frac{p_0}{q_0}} \beta^{p_0-1}\lambda_f(\beta)d\beta \\ &= (q-q_0)^{-\frac{p_0}{q_0}} \int_0^\infty \beta^{p-1}\lambda_f(\beta)d\beta \\ &= (q-q_0)^{-\frac{p_0}{q_0}} p^{-1}||f||_p^p,\end{aligned}$$

and

$$\begin{aligned}\int_0^\infty \left(\int_0^\infty \phi_1(\alpha,\beta)^{\frac{q_1}{p_1}}\,d\alpha\right)^{\frac{p_1}{q_1}} d\beta &= \int_0^\infty \left(\int_{\beta^{1/\sigma}}^\infty \alpha^{q-q_1-1}(\beta^{p_1-1}\lambda_f(\beta))^{\frac{q_1}{p_1}}\,d\alpha\right)^{\frac{p_1}{q_1}} d\beta \\ &= \int_0^\infty \left(\frac{\beta^{(q-q_1)/\sigma}}{q_1-q}\right)^{\frac{p_1}{q_1}} \beta^{p_1-1}\lambda_f(\beta)d\beta \\ &= (q_1-q)^{-\frac{p_1}{q_1}} p^{-1}||f||_p^p.\end{aligned}$$

If $q_0 > q_1$, then we have that $\frac{1}{q_0} < \frac{1}{q} < \frac{1}{q_1}$. As before, we can conclude that

$$\int_0^\infty \left(\int_0^\infty \phi_j(\alpha,\beta)^{\frac{q_j}{p_j}}\,d\alpha\right)^{\frac{p_j}{q_j}} d\beta = |q_j - q|^{-\frac{p_j}{q_j}} p^{-1}||f||_p^p.$$

Using the above, with (1.15) and (1.16), we prove that

$$\sup\{||Tf||_q : ||f|| = 1\} \leq B := 2q^{1/q} \left(\sum_{j=0}^{1} C_j^{q_j} \left(\frac{p_j}{p}\right)^{q_j/p_j} |q_j - p|^{-1} \right)^{1/q}. \tag{1.17}$$

Finally, this implies that $||Tf||_q \leq B||f||_p$.

Third, suppose that $q_0 = \infty$ or $q_1 = \infty$. Here, we may consider three subcases: 1) $p_1 = q_1 = \infty$, 2) $q_0 < q_1 = \infty$ with $p_0 < p_1 < \infty$ and 3) $q_0 < q_1 = \infty$ with $p_1 < p_0 < \infty$.

1) In the case $p_1 = q_1 = \infty$, by assumption $p_0 \leq q_0 < q_1 = \infty$, the clever option of A is α/C_1 since the function h_A satisfies that

$$||Th_A||_\infty \leq C_1||h_A||_\infty \leq \alpha$$

and then $\lambda_{Th_A}(\alpha) = 0$. Using (1.12), $q_0 < q$, $p_0 < p$ and $qp_0 = q_0 p$, we have

$$\begin{aligned}
\int |Tf|^q dx &\leq 2^q q \int_0^\infty \alpha^{q-1} \lambda_{Tg_A}(\alpha) d\alpha \\
&\leq 2^q q \int_0^\infty \alpha^{q-1} (C_0 ||g_A||_{p_0}/\alpha)^{q_0} \\
&\leq 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \alpha^{q-q_0-1} \left(\int_A^\infty \beta^{p_0-1} \lambda_f(\beta) d\beta \right)^{q_0/p_0} d\alpha \\
&\leq 2^q q C_0^{q_0} p_0^{q_0/p_0} \left(\int_0^\infty \left(\int_0^{C_1\beta} \alpha^{q-q_0-1} (\beta^{p_0-1}\lambda_f(\beta))^{\frac{q_0}{p_0}} d\alpha \right)^{\frac{p_0}{q_0}} d\beta \right)^{\frac{q_0}{p_0}} \qquad (1.18) \\
&= 2^q q C_0^{q_0} C_1^{q-q_0} p_0^{q_0/p_0} |q - q_0|^{-1} \left(\int_0^\infty \beta^{p_0-1} \lambda_f(\beta) \beta^{\frac{p_0(q-q_0)}{q_0}} d\beta \right)^{\frac{q_0}{p_0}} \\
&= 2^q q C_0^{q_0} C_1^{q-q_0} \frac{p_0}{p}^{q_0/p_0} |q - q_0|^{-1} ||f||_p^q.
\end{aligned}$$

2) In the case $q_0 < q_1 = \infty$ with $p_0 < p_1 < \infty$, we choose A such that $\lambda_{h_A}(\alpha) = 0$ as well. Observe that in this case,

$$\begin{aligned}
||Th_A||_\infty^{p_1} \leq C_1^{p_1} ||h_A||_{p_1}^{p_1} &= C_1^{p_1} p_1 \int_0^\infty \alpha^{p_1-1} \lambda_{h_A}(\alpha) d\alpha \\
&= C_1^{p_1} p_1 \int_0^A \alpha^{p_1-1} \lambda_f(\alpha) d\alpha \leq C_1^{p_1} \frac{p_1}{p} A^{p_1-p} p \int_0^A \alpha^{p-1} \lambda_f(\alpha) d\alpha \\
&\leq C_1^{p_1} \frac{p_1}{p} A^{p_1-p} ||f||_p^p.
\end{aligned}$$

Therefore if we choose $A = (\alpha/d)^{p_1/(p_1-p)}$ with $d = C_1(p_1||f||_p^p/p)^{p_1}$, we have $||Th_A||_\infty < \alpha$. As before, we have that

$$
\begin{aligned}
\int |Tf|^q dx &\le 2^q q C_0^{q_0} p_0^{q_0/p_0} \int_0^\infty \alpha^{q-q_0-1} \left(\int_A^\infty \beta^{p_0-1} \lambda_f(\beta) d\beta \right)^{q_0/p_0} d\alpha \\
&\le 2^q q C_0^{q_0} p_0^{q_0/p_0} \left(\int_0^\infty \left(\int_0^{d\beta^{\frac{p_1-p}{p_1}}} \alpha^{q-q_0-1} (\beta^{p_0-1}\lambda_f(\beta))^{\frac{q_0}{p_0}} d\alpha \right)^{\frac{p_0}{q_0}} d\beta \right)^{\frac{q_0}{p_0}} \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} \left(\int_0^\infty \left(\int_0^{d\beta^{\frac{p_1-p}{p_1}}} \alpha^{q-q_0-1} d\alpha \right)^{\frac{p_0}{q_0}} \beta^{p_0-1}\lambda_f(\beta) d\beta \right)^{\frac{q_0}{p_0}} \qquad (1.19) \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} d^{q-q_0} (q-q_0)^{-1} \left(\int_0^\infty \beta^{(\frac{p_1-p}{p})(\frac{q-q_0}{q_0})p_0+p_0-1} \lambda_f(\beta) d\beta \right)^{\frac{q_0}{p_0}} \\
&= 2^q q C_0^{q_0} p_0^{q_0/p_0} d^{q-q_0} (q-q_0)^{-1} ||f||_p^{\frac{pq_0}{p_0}}.
\end{aligned}
$$

Where we use the fact that $(\frac{p_1-p}{p_1})(\frac{q-q_0}{q_0})p_0 + p_0 - 1 = p - 1$. This is true since the limit value of (1.13) when $q_1 \to \infty$ says that $\frac{p_1}{p_1-p} = \frac{p_0(q_0-q)}{q_0(p_0-p)}$. We can conclude that

$$
\sup\{||Tf||_q; ||f||_p = 1\} \le B := 2\left(qC_0^{q_0}C_1^{p_1}(p_1/p)^{p_1}(q-q_0)^{-1}\right)^{1/q}
$$

and therefore

$$
||Tf||_q \le B||f||_p.
$$

3) In the last case in which $q_0 < q_1 = \infty$ with $p_1 < p_0 < \infty$, we choose $A = (\alpha/d)^{p_0/(p_0-p)}$ with $d = C_0\left(p_0||f||_p^p/p\right)^{1/p_0}$. The reason for this choice is that by (1.11)

$$
\begin{aligned}
||Tg_A||_\infty^{p_0} &\le C_0^{p_0}||g_A||_{p_0}^{p_0} \\
&\le C_0^{p_0}\frac{p_0}{p}A^{p_0-p}p\int_A^\infty \beta^{p-1}\lambda_f(\beta)d\beta \qquad (1.20) \\
&\le C_0^{p_0}\frac{p_0}{p}A^{p_0-p}||f||_p^p
\end{aligned}
$$

and this choice of A would imply that $\lambda_{Tg_A}(\alpha) = 0$. We omit the rest of the computations, since these are analogous to the subcase 2). □

Chapter 2

On the regularity of the Hardy-Littlewood maximal function

We start the study of the differentiability properties of the maximal function with the theorem presented in the work [7].

2.1 The Maximal Function of a Sobolev Function

Let us start with some basic definitions of Sobolev spaces. We represent with $W^{1,p}(\mathbb{R}^d)$ for $1 \leq p \leq \infty$ the functions in $f \in L^p(\mathbb{R}^d)$ whose weak partial derivatives $D_i f$ belong to $L^p(\mathbb{R}^d)$ for $i = 1, \ldots, d$. We consider $W^{1,p}(\mathbb{R}^d)$ a normed vector space with $||f||_{1,p} = (||f||_p^p + \sum_i ||D_i f||_p^p)^{1/p}$. The idea is to study whether or not the map $f \mapsto Mf$ is bounded from $W^{1,p}(\mathbb{R}^d)$ to $W^{1,p}(\mathbb{R}^d)$. By the comments after Theorem 1.1, we know that the maximal function of an integrable function is never integrable, we have that this is false for $p = 1$.

Theorem 2.1 (Kinnunen [7]). *Let $1 < p < \infty$. If $f \in W^{1,p}(\mathbb{R}^d)$, then $Mf \in W^{1,p}(\mathbb{R}^d)$ and*

$$|D_i Mf| \leq MD_i f, \quad i = 1, \ldots, d, \tag{2.1}$$

almost everywhere in $\mathbb{R}^d$.

An important remark is that (2.1) says that if $f \in W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$,

$$||Mf||_{1,p} \leq A_p ||f||_{1,p}, \tag{2.2}$$

where A_p is the constant of the Theorem 1.1 part iii).

Proof. Let $f \in W^{1,p}(\mathbb{R}^d)$. Observe that if we denote $\chi_r(x) = \frac{\chi_{B(x,r)}}{m(B(x,r))}$ we can rewrite the maximal function as

$$Mf(x) = \sup_{r>0} |f| * \chi_r(x). \tag{2.3}$$

Also, for every $r > 0$, $|f| * \chi_r \in W^{1,p}(\mathbb{R}^d)$ and $D_i(|f| * \chi_r) = \chi_r * D_i|f|$, $i = 1, \ldots, d$. This happens since $|f| \in W^{1,p}(\mathbb{R}^d)$, in fact

$$D_i|f|(x) = \begin{cases} D_i f & \text{if } f > 0, \\ 0 & \text{if } f = 0, \\ -D_i f & \text{if } f < 0 \end{cases} \tag{2.4}$$

(see Lemma 7.5 in [6]) and

$$\begin{aligned}
\int_{\mathbb{R}^d} (|f| * \chi_r)(x)\phi'(x)dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y-x)|\chi(y)dy\, \phi'(x)\, dx \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(y-x)|\phi'(x)\, dx\, \chi(y)\, dy \\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} D_i|f(y-x)|\phi(x)\, dx\chi(y)\, dy \\
&= \int_{\mathbb{R}^d} D_i|f| * \chi(x)\phi(x)\, dx,
\end{aligned}$$

for every differentiable function $\phi \in C_c^\infty(\mathbb{R}^d)$. By Young's inequality and (2.4), $||D_i(|f| * \chi_r)||_p \leq ||D_i|f|||_p = ||D_i f||_p$.

We can restrict the supremum of the definition of the maximal function in (2.3) to positive rationals. The reason is that as a function of r,

$$|f| * \chi_r(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|dy$$

is the product of two continuous functions. The Lebesgue measure is continuous and f is locally integrable. Now, let $\{r_n\}$ be any enumeration of the positive rational numbers, and define

$$g_m(x) := \max_{1 \leq n \leq m} |f| * \chi_{r_n}(x). \tag{2.5}$$

Clearly g_m is a nondecreasing sequence of functions, in the sense $g_{m_1}(x) \leq g_{m_2}(x)$ for any $m_1, m_2 \in \mathbb{N}$, $m_1 \leq m_2$ and for any $x \in \mathbb{R}^d$ and

$$||g_m||_p \leq ||Mf||_p. \tag{2.6}$$

We prove that $g_m(x) \to Mf(x)$ as $m \to \infty$ for almost every $x \in \mathbb{R}^d$. Suppose that $Mf(x) < \infty$ and fix $\varepsilon > 0$, there exists a rational r_N such that

$$Mf(x) - \varepsilon < |f| * \chi_{r_N}(x),$$

we can conclude that

$$|Mf(x) - g_m(x)| = Mf(x) - g_m(x) < \varepsilon \tag{2.7}$$

for every $m \geq N$. Finally, by Theorem 1.1 part i), $m\{x; Mf(x) = \infty\} = 0$. In conclusion, this pointwise convergence holds up to a set of measure zero of $\mathbb{R}^d$.

Now notice that for any pair of functions g, h, from the equality

$$\max\{g(x), h(x)\} = \frac{1}{2}(g(x) + h(x) + |g(x) - h(x)|)$$

and (2.4) we can show that $\{g_m\}$ is a sequence of functions in $W^{1,p}(\mathbb{R}^d)$. In addition, we can also observe that for every $m \geq 1$

$$\begin{aligned} |D_i g_m(x)| &\leq \max_{1 \leq n \leq m} D_i(|f| * \chi_{r_n})(x) \\ &= \max_{1 \leq n \leq m} \chi_{r_n} * D_i|f|(x) \\ &\leq MD_i|f|(x) = MD_i f(x), \end{aligned} \tag{2.8}$$

for $i = 1, \ldots, d$ and almost every $x \in \mathbb{R}^d$. Therefore

$$||D_i g_m||_p \leq ||MD_i f||_p \leq A_p ||D_i f||_p \tag{2.9}$$

using Theorem 1.1 part iii). The inequalities (2.6) and (2.9) imply that for $m = 1, 2, \ldots$

$$\begin{aligned} ||g_m||_{1,p} &= ||g_m||_p + \sum_{i=1}^{d} ||D_i g_m||_p \\ &\leq A_p \left(||f||_p + \sum_{i=1}^{d} ||D_i f||_p \right) \\ &= A_p ||f||_{1,p} < \infty. \end{aligned} \tag{2.10}$$

Hence, $\{g_m\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^d)$ which converges pointwise to Mf almost everywhere. The space $W^{1,p}(\mathbb{R}^d)$ is reflexive for $1 < p < \infty$ and the weak compactness of $W^{1,p}(\mathbb{R}^d)$ implies that there exists a subsequence denoted by $\{g_m\}$

as well, that converges weakly to some $g \in W^{1,p}(\mathbb{R}^d)$. We prove that $Mf = g$ almost everywhere. Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a smooth function of compact support, then

$$\begin{aligned} \left|\int (g - Mf)\phi dx\right| &\leq \left|\int (g - g_m)\phi dx\right| + \left|\int (g_m - Mf)\phi dx\right| \\ &\leq \left|\int (g - g_m)\phi dx\right| + ||g_m - Mf||_p ||\phi||_{p'} =: I_m + J_m \end{aligned} \tag{2.11}$$

for every $m = 1, 2, \ldots$. On one hand g_m converges weakly to g, hence $I_m \to 0$. On the other hand, $J_m \to 0$ using the Dominated Convergence Theorem. This implies that $\int (g - Mf)\phi dx = 0$. Since ϕ is arbitrary, it follows that $Mf = g$ almost everywhere in $\mathbb{R}^d$ and therefore $Mf \in W^{1,p}(\mathbb{R}^d)$. This also implies that $D_i g_m$ converges weakly in $L^p(\mathbb{R}^d)$ to $D_i Mf$ because, for every $\phi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned} \int D_i Mf \phi dx &= -\int Mf D_i \phi dx \\ &= -\lim_{m\to\infty} \int g_m D_i \phi dx = \lim_{m\to\infty} \int D_i g_m \phi dx. \end{aligned} \tag{2.12}$$

This weak convergence also implies that for every $i = 1, \ldots, d$

$$|D_i Mf| \leq M D_i f \tag{2.13}$$

almost everywhere in $\mathbb{R}^d$. Suppose that this is not true, and $B = \{x; |D_i Mf| > M D_i f\}$ has positive measure for some i; for our purpose we can assume it has finite measure. The characteristic function χ_B belongs to L^q (with q such that $q^{-1} + p^{-1} = 1$). Hence,

$$\lim_{m\to\infty} \int |D_i g_m| \chi_B dx = \int |D_i Mf| \chi_B dx > \int M D_i f \chi_B dx$$

which is impossible since (2.8) implies $\int |D_i Mf| \chi_B dx \leq \int M D_i f \chi_B dx$. □

There is another way to prove the last part of this theorem. Certainly, $D_i g_m$ converges weakly in $L^p(\mathbb{R}^d)$ to $D_i Mf$, Mazur's corollary (See for instance [2], Corollary 3.8) tell us that there exists a sequence made up of convex combinations of $\{D_i g_m\}$ which converges to $D_i Mf$ in norm $L^p(\mathbb{R}^d)$. Consequently, a subsequence of these convex combinations converges pointwise almost everywhere to $D_i Mf$, we are done because this subsequence is dominated by $M D_i f$ by inequality (2.8).

The previous theorem also holds for $p = \infty$, the proof is very simple using a characterizations of Sobolev spaces given in Proposition 9.3 of [2]. Let $f \in W^{1,\infty}$, $h \in \mathbb{R}^d$ and denote $\tau_h(Mf)(x) := Mf(x+h)$. Hence

$$||\tau_h(Mf) - Mf||_\infty \leq ||M(\tau_h f - f)||_\infty \leq ||\tau_h f - f||_\infty \leq C|h|$$

where we have used the sublinearity of the maximal function and part iii) of Theorem 1.1.

2.2 Application: capacity and quasicontinuity

In [7], the author noticed that Theorem 2.1 implies immediately a weak type of inequality for the Sobolev capacity. First, let us define the **Sobolev p-capacity** of a set $E \subset \mathbb{R}^d$ as

$$C_p(E) = \inf_{f \in \mathcal{A}(E)} \int_{\mathbb{R}^d} |f|^p + \sum_{i=1}^{d} |D_i f|^p dx$$

where $\mathcal{A}(E)$ is a collection of functions

$$\mathcal{A}(E) = \{f \in W^{1,p}(\mathbb{R}^d); f \geq 1 \text{ on a neighboorhood of } E\}.$$

If this class of functions is empty for a set E, the p-capacity is defined as ∞. Also, functions on $\mathcal{A}(E)$ are called admissible functions for E. We mention some interesting properties:

- The Sobolev p-capacity is an outer measure: it is monotone and countably subadditive.
- It is outer regular: $C_p(E) = \inf\{C_p(U); E \subset U, U \text{ is open}\}$.
- The inequality $m(E) \leq C_p(E)$ always holds.

For more properties see references in [7].

Let $f \in W^{1,p}(\mathbb{R}^d)$, and let λ be a positive number. Denote

$$E_\lambda = \{x; Mf(x) > \lambda\}.$$

By Theorem 2.1 the function $Mf/\lambda \in W^{1,p}(\mathbb{R}^d)$ is admissible for the set E_λ. The inequality (2.13) implies that

$$
\begin{aligned}
C_p(E_\lambda) &\leq \frac{1}{\lambda^p}\int_{\mathbb{R}^d} |Mf|^p + \sum_{i=1}^d |D_i Mf|^p dx \\
&\leq \frac{1}{\lambda^p}\int_{\mathbb{R}^d} |Mf|^p + \sum_{i=1}^d (MD_i f)^p dx \\
&\leq \frac{A_p^p}{\lambda^p}\left(||f||_p^p + \sum_{i=1}^d ||D_i f||_p^p \right) \\
&= \frac{A_p^p ||f||_{1,p}^p}{\lambda^p}.
\end{aligned}
\tag{2.14}
$$

We will say that a property holds p**-quasieverywhere** if it is true, except on a set of Sobolev p-capacity zero. It is an analogous property to the Lebesgue almost everywhere. Now, a function f is p**-quasicontinuous** in $\mathbb{R}^d$ if for every $\varepsilon > 0$ there is a set F such that $C_p(F) < \varepsilon$ and f is continuous and finite in the complement of F.

There are some properties of p-quasicontinuity that are very useful, for a proof see the references in [7].

- For each $f \in W^{1,p}(\mathbb{R}^d)$ there is a p-quasicontinuous representative. This means that exists a p-quasicontinuous function $\tilde{f} \in W^{1,p}(\mathbb{R}^d)$ with $f = \tilde{f}$ almost everywhere.

- If $f, g \in W^{1,p}(\mathbb{R}^d)$ are p-quasicontinuous functions and $f = g$ almost everywhere, then $f = g$ p-quasieverywhere.

- The p-quasicontinuous representative of a Sobolev function is unique in the sense considered above.

Theorem 2.2. *If $f \in W^{1,p}(\mathbb{R}^d)$, $1 < p < \infty$, then Mf is p-quasicontinuous.*

Proof. First, we prove that $Mf \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ if $f \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. Let

$x, h \in \mathbb{R}^d$ and $\varepsilon > 0$, then there exists $r_\varepsilon > 0$ such that

$$\begin{aligned}
&\frac{1}{m(B(x,r))}\int_{B(x,r)} |(\tau_h f)(y) - f(y)|\, dy \\
&\qquad \le \left(\frac{1}{m(B(x,r))}\int_{B(x,r)} |(\tau_h f)(y) - f(y)|^p\, dy\right)^{1/p} \\
&\qquad \le \frac{||\tau_h f - f||_p}{m(B(x,r))^p} \\
&\qquad \le \frac{2||f||_p}{m(B(x,r))^p} \\
&\qquad < \varepsilon,
\end{aligned}$$

for every $r \ge r_\varepsilon$. Notice that we have used the Young's inequality in the first inequality. If $r < r_\varepsilon$, there exist $\delta > 0$ such that

$$\frac{1}{m(B(x,r))}\int_{B(x,r)} |(\tau_h f)(y) - f(y)|\, dy \le \sup_{B(x,r_\varepsilon)} |\tau_h f - f| < \varepsilon$$

for every $|h| < \delta$. We deduce that

$$|\tau_h(Mf)(x) - Mf(x)| \le M(\tau_h f - f)(x) \le \varepsilon$$

for every $|h| < \delta$. Then $Mf \in C(\mathbb{R}^d)$, the Theorem 1.1 proves that $Mf \in L^p(\mathbb{R}^d)$.

Let $f \in W^{1,p}(\mathbb{R}^d)$ and a sequence of functions $\{\varphi_i\}$ with each $\varphi_i \in C_0^\infty(\mathbb{R}^d)$ and $\varphi_i \to f$ in $W^{1,p}(\mathbb{R}^d)$. By the weak inequality (2.14) there is a set F of capacity zero, $C_p(F) = 0$ and Mf is finite in the complement $\mathbb{R}^d \backslash F$. Denote again $\{\varphi_i\}$ a subsequence, such that

$$||\varphi_i - f||_{1,p}^p \le (4^i A_p)^{-p}.$$

This implies that on the set $E_i = \{x \in \mathbb{R}^d \backslash F;\, M(\phi_i - f)(x) > 2^{-i}\}$, for $i = 1, 2, \ldots$. Using one more time the weak inequality (2.14) we have

$$C_p(E_i) \le 2^{ip} A_p^p ||\varphi_i - f||_{1,p}^p \le 2^{-ip}.$$

Using the subadditivity property of the p-capacity, for

$$F_j = \cup_{i \ge j} E_i$$

we have that

$$C_p(F_j) \le \sum_{i \ge j} 2^{-ip} < \infty.$$

This proves that $C_p(F_j) \to 0$ as $j \to \infty$. The convergence $M\varphi_i \to Mf$ is uniform in $\mathbb{R}^d \backslash F_j$ due to

$$|M\varphi_i(x) - Mf(x)| \leq M(\varphi_i - f)(x) \leq 2^{-i}$$

for every $x \in \mathbb{R}^d \backslash F_j$ and $i \geq j$. As the uniform limit of continuous functions is continuous, Mf is continuous in $\mathbb{R}^d \backslash F_j$. We are done if we let $F = \cap_{j \geq 1} F_j$ in the definition of p-quasicontinuity. $\square$

Chapter 3

The regularity at $p = 1$

Despite the fact that M does not map $W^{1,1}(\mathbb{R}^d)$ to itself since Mf is not even in $L^1(\mathbb{R}^d)$, Tanaka's Theorem [12] gives an interesting alternative that we want to present. Tanaka proved that if $f \in W^{1,1}(\mathbb{R})$, the $D\widetilde{M}f \in L^1(\mathbb{R})$ and $||D\widetilde{M}f||_1 \leq c||Df||_1$ for $c = 2$. This inequality is an analog to the result of Kinnunen, observe that (2.1) and Theorem 1.1 part iii) implies $||DMf||_p \leq A_p||Df||_p$ for $1 < p < \infty$.

3.1 Tanaka's theorem

As Tanaka remarks in his paper [12], Theorem 2.1 can be also proved for the non-centered maximal operator. Instead of (2.5), we can take the sequence

$$g_m(x) = \max_{\substack{1 \leq n \leq m \\ x \in B_n}} |f| * \chi_{B_m}(x), \tag{3.1}$$

where χ is the indicator function of the ball B_m normalized by the Lebesgue measure. The collection $\{B_m\}$ is an enumeration of balls centered in a countable dense subset of $\mathbb{R}^d$ and positive rational radius.

Theorem 3.1 (Tanaka [12])**.** *If $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{M}f$ has a weak derivative and it is an integrable function. Moreover,*

$$||(\widetilde{M}f)'||_1 \leq 2||f'||_1.$$

We prove some propositions before going to the proof of the Theorem 3.1.

Proposition 3.2. *Let $f \in W^{1,1}(\mathbb{R})$, then $\widetilde{M}f$ is bounded.*

Proof. Without loss of generality, we can assume f is absolutely continuous function, see Theorem 8.2 in [2]. Since it is differentiable with derivative f' in $L^1(\mathbb{R})$, f must be bounded.

$$|f(x)| \leq |f(0)| + \int_0^x |f'(y)|dy \leq |f(0)| + ||f'||_1 \leq c < \infty.$$

By Theorem 1.1 part iii) the conclusion follows, since $\widetilde{M}f(x) \leq c$. □

Definition. For a locally integrable function f on $\mathbb{R}$, define the one-sided maximal operator functions $M_l f$ and $M_r f$ by

$$M_l f(x) = \sup_{s>0} \frac{1}{s} \int_{x-s}^{x} |f(y)|dy,$$
$$M_r f(x) = \sup_{t>0} \frac{1}{t} \int_{x}^{x+t} |f(y)|dy.$$

Proposition 3.3. *Let $f \in L^1_{loc}(\mathbb{R})$. Then*

$$\widetilde{M}f(x) = \max\{M_l f(x), M_r f(x)\}.$$

Proof. On one hand, we have the inequalities

$$\widetilde{M}f(x) \geq M_l f(x) \quad \text{and} \quad \widetilde{M}f(x) \geq M_r f(x).$$

It implies $\widetilde{M}f(x) \geq \max\{M_l f(x), M_r f(x)\}$. On the other hand for $s, t > 0$,

$$\begin{aligned}\frac{1}{s+t}\int_{x-s}^{s+t} |f(y)|dy &= \frac{s}{s+t}\frac{1}{s}\int_{x-s}^{x} |f(y)|dy + \frac{t}{s+t}\frac{1}{t}\int_{x}^{x+t} |f(y)|dy \\ &\leq \frac{s}{s+t} M_l f(x) + \frac{t}{s+t} M_r f(x) \\ &\leq \max\{M_l f(x), M_r f(x)\},\end{aligned}$$

which implies $\widetilde{M}f(x) \leq \max\{M_l f(x), M_r f(x)\}$. □

Proposition 3.4. *Let $f \in W^{1,1}(\mathbb{R})$. Then $M_l(f)$ and $M_r(f)$ are continuous and vanishes at infinity.*

Proof. It is enough to prove the result for $M_r(f)$ since the same reasoning can be applied to $M_l(f)$. If $f \in W^{1,1}(\mathbb{R})$, we can assume it to be continuous, and therefore uniformly continuous, see Theorem 8.2 in [2]. Let $x, h \in \mathbb{R}$ and $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that

$$\frac{1}{r}\int_x^{x+r} |f_h(y) - f(y)|dy \leq \frac{2||f||_1}{r} < \varepsilon$$

if $r \geq r_\varepsilon$. Since f is uniformly continuous, for $0 < r \leq r_\varepsilon$, there exists $\delta > 0$ such that $|f(y_1) - f(y_2)| < r\varepsilon$ if $|f(y_1) - f(y_2)| < \delta$. Then

$$\frac{1}{r}\int_x^{x+r} |f_h(y) - f(y)|dy < \varepsilon, \quad \text{for } |h| < \delta.$$

This implies that $M_r(f_h - f)(x) \leq \varepsilon$ for $|h| < \delta$ and

$$|(\tau_h(M_r f)(x) - M_r f(x)| \leq M_r(\tau_h f - f)(x) \leq \varepsilon.$$

In conclusion $M_r f$ is continuous.

We will prove that $M_r f(x) \to 0$ if $x \to -\infty$. Proving that $M_r f(x) \to 0$ if $x \to +\infty$ require minor modifications to our argument. Let $\varepsilon > 0$, since $||f||_1 < \infty$ there exists $R_\varepsilon > 0$ such that $\frac{||f||_1}{t} < \frac{\varepsilon}{2}$ if $t \geq R_\varepsilon$. Since f vanishes at infinity, there exists $Q_\varepsilon > 0$ such that $|f(y)| < \frac{\varepsilon}{2}$ if $|y| \geq Q_\varepsilon$. Consider $x < -(Q_\varepsilon + R_\varepsilon)$, there exist $t > 0$ such that

$$M_r f(x) - \frac{\varepsilon}{2} < \frac{1}{t}\int_x^{x+t} |f(y)|dy.$$

We have to consider the following two cases

Case (i) $x + t \leq -Q_\varepsilon$. We have that $x \leq y \leq x + t$, and therefore on this interval $|y| \geq Q_\varepsilon$, this implies that $M_r f(x) < \varepsilon$.

Case (ii) $x + t > -Q_\varepsilon$. We have chosen $-x > Q_\varepsilon + R_\varepsilon$ then $t > -Q_\varepsilon - x > R_\varepsilon$. Therefore,

$$\frac{1}{t}\int_x^{x+t} |f(y)|dy < \frac{||f||_1}{t} < \frac{\varepsilon}{2}$$

and also $M_r f(x) < \varepsilon$. □

Proposition 3.4 has two relevant consequences for the proof of the Theorem 3.1 as we will see. First, combining with Proposition 3.3, $\widetilde{M}f$ is continuous. Second, the set

$$E = \{x \in \mathbb{R}; M_l f(x) > |f(x)|\}$$

is open and so E can be written as countably union of disjoint intervals, say

$$E = \cup_j I_j = \cup_j (\alpha_j, \beta_j).$$

Lemma 3.5. *Using the notation above, the following facts hold*

(i) $M_l f$ *is a non-increasing function on each* I_j.

(ii) $M_l f$ *is a locally Lipschitz function on each* I_j. *In particular,* $M_l f$ *is an absolutely continuous function on each compact subinterval of* I_j.

Proof. Part (i). Let $K = [\alpha, \beta] \subset I_j$. It suffices to prove that $M_l f$ is nonincreasing on K. We know that $M_l f(x) - f(x) > 0$ on K and the continuity of $M_l f - |f|$ implies

$$\varepsilon := \min_{x \in K} M_l f(x) - |f(x)| > 0. \tag{3.2}$$

By the uniform continuity of $|f|$ on $\mathbb{R}$ there exists $\delta > 0$ such that

$$|f(y)| < |f(x)| + \frac{\varepsilon}{2} \quad \text{for all } x \in K, \quad |y - x| \leq \delta. \tag{3.3}$$

The definition of ε and (3.3) imply that

$$M_l f(x) = \sup_{s > \delta} \frac{1}{s} \int_{x-s}^{x} |f(y)| dy, \quad \text{for all } x \in K. \tag{3.4}$$

For if $M_l f(x) = \sup_{s \leq \delta} \frac{1}{s} \int_{x-s}^{x} |f(y)| dy$ for some $x \in K$ then using (3.3) we deduce that $M_l f(x) \leq |f(x)| + \frac{\varepsilon}{2}$ and then a contradiction to the definition of ε. We will prove that

$$M_l f(x - h) \geq M_l f(x) \quad \text{for } x - h, x \in K,\ 0 < h \leq \delta. \tag{3.5}$$

Suppose that $s > \delta$, x and h as in (3.5). By the above, we have the following

$$\begin{aligned} \frac{1}{s} \int_{x-s}^{x} |f(y)| dy &= \frac{s-h}{s} \left\{ \frac{1}{s-h} \int_{x-s}^{x-h} |f(y)| dy \right\} + \frac{h}{s} \left\{ \frac{1}{h} \int_{x-h}^{x} |f(y)| dy \right\} \\ &\leq \frac{s-h}{s} M_l f(x-h) + \frac{h}{s} \left\{ |f(x)| + \frac{\varepsilon}{2} \right\} \\ &\leq \max \left\{ M_l f(x-h), |f(x)| + \frac{\varepsilon}{2} \right\}. \end{aligned}$$

From the above and (3.4), we deduce that

$$M_l f(x) \leq \max \left\{ M_l f(x-h), |f(x)| + \frac{\varepsilon}{2} \right\}.$$

Since the definition of ε implies that $M_l f(x) > |f(x)| + \varepsilon$ we have (3.5).

Part (ii). Let K as in part (i) and δ such that (3.3) holds. Now suppose that $x, x+h \in K$, $h > 0$, and $s > \delta$. From part (i)

$$\begin{aligned}
\frac{1}{s}\int_{x-s}^{x} |f(y)|dy - M_l f(x+h) &\le \frac{1}{s}\int_{x-s}^{x} |f(y)|dy - \frac{1}{s+h}\int_{x-s}^{x+h} |f(y)|dy \\
&\le \frac{1}{s}\int_{x-s}^{x} |f(y)|dy - \frac{1}{s+h}\int_{x-s}^{x} |f(y)|dy \\
&= \frac{h}{s+h}\frac{1}{s}\int_{x-s}^{x} |f(y)|dy \le \frac{M_l f(x)}{\delta} h \\
&\le \frac{M_l f(\alpha)}{\delta} h.
\end{aligned}$$

Now, taking the supremum on the left side of above when $s > \delta$, we obtain

$$0 \le M_l f(x) - M_l f(x+h) \le Ch.$$

□

Proposition 3.6. *If $f \in W^{1,1}(\mathbb{R})$, then $M_l f$ and $M_r f$ are weakly differentiable and the weak derivatives are integrable functions. Moreover,*

$$||(M_l f)'||_1 \le ||f'||_1, \quad ||(M_r f)'||_1 \le ||f'||_1.$$

Proof. We prove the result only for $M_l f$ since it is analogous for $M_r f$. We can write

$$|f|' = \begin{cases} f' & \text{if } f > 0, \\ 0 & \text{if } f = 0, \\ -f' & \text{if } f < 0 \end{cases}$$

almost everywhere in $\mathbb{R}$, therefore, $|||f|'||_1 = ||f'||_1$. Observe that on every $(\alpha_j, \beta_j) \subset E$, $M_l f$ is differentiable almost everywhere by Lemma 3.5, let us denote v such function which satisfies $v \le 0$ on E. On the set $F := \mathbb{R}\backslash E$, by continuity $M_l f = |f|$, and so we prove that

$$(M_l f)' = \chi_E v + \chi_F |f|'. \tag{3.6}$$

Let $\phi \in C_c^\infty(\mathbb{R})$, we will prove that:

$$\int_{I_j} M_l f(y)\phi'(y)dy = |f(\beta_j)|\phi(\beta_j) - |f(\alpha_j)|\phi(\alpha_j) - \int_{I_j} v(y)\phi(y)dy. \tag{3.7}$$

It is possible that $\alpha_j = -\infty$ or $\beta_j = \infty$, but we know that f vanishes at infinity and Proposition 3.4 holds, so we let $M_l(\alpha_j) = M_l(\beta_j) = 0$ and $f(\alpha_j) = f(\beta_j) = 0$. Using Lebesgue dominated theorem

$$\begin{aligned}
\int_{\alpha_j}^{\beta_j} M_l f(y)\phi'(y)dy &= \lim_{h\to 0}\int_{\alpha_j}^{\beta_j} \frac{M_l f(y)\phi(y+h) - M_l f(y)\phi(y)}{h}dy \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\int_{\alpha_j+h}^{\beta_j+h} M_l f(y-h)\phi(y)dy - \int_{\alpha_j}^{\beta_j} M_l f(y)\phi(y)dy\right\} \\
&= \lim_{h\to 0}\frac{1}{h}\left\{\int_{\beta_j}^{\beta_j+h} M_l f(y-h)\phi(y)dy - \int_{\alpha_j}^{\alpha_j+h} M_l f(y-h)\phi(y)dy\right\} \\
&\quad - \lim_{h\to 0}\int_{\alpha_j}^{\beta_j} \frac{\{M_l f(y-h) - M_l f(y)\}\,\phi(y)}{-h}dy \\
&= Mf(\beta_j)\phi(\beta_j) - |M(\alpha_j)|\phi(\alpha_j) - \int_{I_j} v(y).
\end{aligned}$$

Equation (3.7) follows, since $Mf(\alpha_j) = |f(\alpha_j)|$ and $Mf(\beta_j) = |f(\beta_j)|$. Using (3.7), we can deduce

$$\begin{aligned}
\int_{\mathbb{R}} M_l f(y)\phi'(y)dy &= \int_E M_l f(y)\phi'(y)dy + \int_F M_l f(y)\phi'(y)dy \\
= \sum_j |f(\beta_j)|\phi(\beta_j) - |f(\alpha_j)|\phi(\alpha_j) &- \int_E v(y)\phi(y)dy + \int_F M_l f(y)\phi'(y)dy \\
&= \int_E \{|f(y)|\phi(y)\}'\,dy - \int_E v(y)\phi(y)dy + \int_F |f(y)|\phi'(y)dy \\
&= \int_E |f(y)|'\phi(y)dy - \int_E v(y)\phi(y)dy + \int_{\mathbb{R}} |f(y)|\phi'(y)dy \\
&= \int_E |f(y)|'\phi(y)dy - \int_E v(y)\phi(y)dy - \int_{\mathbb{R}} |f(y)|'\phi(y)dy \\
&= -\int_F |f(y)|'\phi(y)dy - \int_E v(y)\phi(y)dy \\
&= -\int_{\mathbb{R}} \left(\chi_E(y)v(y) + \chi_F(y)|f|'(y)\right)\phi(y)dy,
\end{aligned}$$

which proves (3.6).

On each interval I_j the weak derivative v is non-positive, so we have

$$\begin{aligned}\int_{I_j} |v(y)|dy &= M_l f(\alpha_j) - M_l f(\beta_j) = |f(\alpha_j)| - |f(\beta_j)| \\ &= -\int_{I_j} |f(y)|' dy \leq \int_{I_j} \left||f(y)|'\right| dy.\end{aligned}$$

We obtain the result since

$$\begin{aligned}||(M_l f)'||_1 &= \int_{\mathbb{R}} |\chi_E(y)v(y) + \chi_F(y)|f|'(y)|dy \\ &= \int_E |v(y)|dy + \int_F ||f|'(y)|dy \\ &\leq \int_E \left||f(y)|'\right| dy + \int_F ||f|'(y)|dy \\ &= ||f'||_1.\end{aligned}$$

□

Lemma 3.7. *Let f and g be integrable functions and set $F(x) = \int_{-\infty}^x f(y)dy$, $G(x) = \int_{-\infty}^x g(y)dy$ and $H(x) = \max\{F(x), G(x)\}$. Then the weak derivative of H is an integrable function, and*

$$||H'||_1 \leq ||f||_1 + ||g||_1$$

Proof. The result follows from the equation

$$\max\{F(x), G(x)\} = \frac{1}{2}\left\{F(x) + G(x) + |F(x) - G(x)|\right\}$$

and the chain rule for weak derivatives. □

Proof theorem 3.1. Combining Proposition 3.3, Lemma 3.7 and Proposition 3.6 we obtain that Mf has an integrable weak derivative and $||(Mf)'||_1 \leq ||(M_l f)'||_1 + ||(M_r f)'||_1 \leq 2||f'||_1$. □

Chapter 4

Discrete analogues of the maximal function

What we have reviewed in the precedent chapters can be also translated in a discrete setting. For simplicity, we will only present the one dimensional case on the discrete maximal operator.

We will present in this chapter two theorems of the work [1]. The first shows that for a function of bounded variation f, the non-centered maximal function $\widetilde{M}f$ has also bounded variation and $\mathrm{Var}(\widetilde{M}f) \leq \mathrm{Var}(f)$ and the second theorem states that for a function in ℓ^1, the centered maximal function Mf has bounded variation controlled by f as $\mathrm{Var}(Mf) \leq c||f||_{\ell^1}$. We use the ideas in [8] to show that $c = 2$ is the lowest positive number that can be used in this inequality. Formally, we want to show, for the non-centered maximal function, the following theorem.

Theorem 4.1. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function of bounded variation. Then*

$$\mathrm{Var}(\widetilde{M}f) \leq \mathrm{Var}(f),$$

and the inequality is sharp.

While for the centered maximal function, we want to show the following

Theorem 4.2. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a function in $\ell^1(\mathbb{Z})$. Then*

$$\mathrm{Var}(Mf) \leq 2||f||_{\ell^1(\mathbb{Z})},$$

and the inequality is sharp.

4.1 The discrete one-dimensional setting

Let $f : \mathbb{Z} \to \mathbb{R}$ a discrete function and let $\mathbb{Z}^+ = \{0, 1, \dots\}$. The discrete **centered** Hardy-Littlewood maximal function is defined by

$$Mf(n) = \sup_{r\in\mathbb{Z}^+} \frac{1}{2r+1} \sum_{k=-r}^{k=r} |f(n+k)|\,,$$

while the **non-centered** version is defined by

$$\widetilde{M}f(n) = \sup_{r,s\in\mathbb{Z}^+} \frac{1}{s+r+1} \sum_{k=-r}^{k=s} |f(n+k)|\,.$$

We establish the following conventions. For $1 \le p < \infty$, the ℓ^p-norm of a function $f : \mathbb{Z} \to \mathbb{R}$ is

$$||f||_{\ell^p(\mathbb{Z})} = \left(\sum_{n=-\infty}^{\infty} |f(n)|^p \right)^{1/p},$$

and ℓ^∞-norm

$$||f||_{\ell^\infty(\mathbb{Z})} = \sup_{n\in\mathbb{Z}} |f(n)|.$$

Consequently, the space $\ell^p(\mathbb{Z})$ consists of the functions f defined on $\mathbb{Z}$ with values on $\mathbb{R}$ such that $||f||_{\ell^p(\mathbb{Z})} < \infty$.

We define the **derivatives** of a discrete function by

$$\begin{aligned}
f'(n) &= f(n+1) - f(n),\\
f''(n) &= f(n+2) - 2f(n+1) + f(n),\\
f'''(n) &= f(n+3) - 3f(n+2) + 3f(n+1) - f(n)
\end{aligned}$$

and so on. If $f \in \ell^p(\mathbb{Z})$, it is possible to check that for any $k \ge 1$,

$$||f^{(k)}||_{\ell^p(\mathbb{Z})} \le 2^k ||f||_{\ell^p(\mathbb{Z})}$$

using the Binomial theorem and Jensen's inequality. This means that the analogous Sobolev spaces $w^{k,p}(\mathbb{Z})$ are again the spaces ℓ^p.

Let $f : \mathbb{Z} \to \mathbb{R}$. The **total variation** of f is given by

$$\mathrm{Var}(f) = ||f'||_{\ell^1(\mathbb{Z})} = \sum_{n=-\infty}^{\infty} |f(n+1) - f(n)|.$$

Example 4.1. Consider the function

$$f(n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that

$$f'(n) = \begin{cases} -1 & \text{if } n = 0, \\ 1 & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$

and so $\mathrm{Var}(f) = 2$. Now observe that for $s, r \in \mathbb{Z}^+$,

$$\frac{1}{r+s+1} \sum_{k=-r}^{k=s} |f(n+k)| = \begin{cases} \frac{1}{r+s+1} & \text{if } n-r \leq 0 \leq n+s, \\ 0 & \text{otherwise.} \end{cases}$$

A short analysis says that both the non-center and center maximal operators attain its value and

$$\begin{aligned} Mf(n) &= \frac{1}{2|n|+1}, \\ \widetilde{M}f(n) &= \frac{1}{|n|+1}. \end{aligned}$$

It is easy to check that $\mathrm{Var}(Mf) = \mathrm{Var}(\widetilde{M}f) = 2$. This example motivates the following:

Proposition 4.3. *Let $f : \mathbb{Z} \to \mathbb{R}$. If $f \in \ell^1(\mathbb{Z})$, then the centered (non-centered) maximal operator attains its value.*

Proof. The idea of the proof is based on the fact that $f(n) \to 0$ when $|n| \to \infty$. Let $n \in \mathbb{Z}$, we want to proof $Mf(n) = \frac{1}{2r+1} \sum_{k=-r}^{k=r} |f(n+k)|$ for some $r \in \mathbb{Z}^+$. Because of translations, we can assume $n = 0$. Fixed $\varepsilon > 0$ there exists $N \in \mathbb{Z}^+$ such that $|f(n)| < \varepsilon$ for $|n| > N$. For every $m > N$, the average can be decomposed as

$$\frac{1}{2m+1} \sum_{k=-m}^{k=m} |f(k)| \leq \frac{1}{2m+1} \sum_{k=-N}^{k=N} |f(k)| + \varepsilon \frac{2(m-N)}{2m+1},$$

we notice that the averages goes to zero as $m \to 0$. Clearly, there must exists $r \in \mathbb{Z}^+$ for which $\frac{1}{2r+1} \sum_{k=-r}^{k=r} |f(k)|$ equals $Mf(0)$. A similar analysis proves that result for the non-centered maximal operator. $\square$

4.1.1 On the non-centered discrete maximal function

We will say that a point n is a local maximum of f if

$$f(n-1) \leq f(n) \quad \text{and} \quad f(n+1) < f(n)$$

and similarly, we will say that a point n is a local minimum of f if

$$f(n-1) \geq f(n) \quad \text{and} \quad f(n+1) > f(n).$$

Lemma 4.4. *Let $f : \mathbb{Z} \to \mathbb{R}$ be a bounded function. If n is a local maximum of $\widetilde{M}f$, then $\widetilde{M}f(n) = |f(n)|$.*

Proof. Without loss of generality, we can assume that f is non-negative. We prove this lemma by contradiction, we let n to be a local maximum and $\widetilde{M}f(n) > f(n)$

Case 1. If $\widetilde{M}f(n)$ is equal to the average on the interval $[n-r, n+s]$ for some $r, s \in \mathbb{Z}^+$. This interval cannot be degenerated by assumption, so it may contain $n+1$ or $n-1$. In the former, we have the inequality

$$\widetilde{M}f(n) = \frac{1}{r+s+1} \sum_{k=-r}^{k=s} f(n+k) = \frac{1}{r+s+1} \sum_{k=-r-1}^{k=s-1} f(n+1+k) \leq \widetilde{M}f(n+1)$$

which is a contradiction. In the later, if the interval does not contain $n+1$, we must have $s = 0$. Taking an average on $[n-r, n-1]$ for $n-1$ we have

$$\widetilde{M}f(n-1) \geq \frac{1}{r} \sum_{k=-r+1}^{k=0} f(n-1-k) = \frac{1}{r} \sum_{k=-r}^{k=-1} f(n-k)$$

and then

$$\widetilde{M}f(n) \leq (r+1)\widetilde{M}f(n) - r\widetilde{M}f(n-1) \leq f(n)$$

Case 2. If $\widetilde{M}f(n)$ is not attained for any $r, s \in \mathbb{Z}^+$. In such case, we can prove that $\widetilde{M}f(m) \geq \widetilde{M}f(n)$ for every $m \in \mathbb{Z}$, therefore, n cannot be a local maximum. Let $c > 0$, such that $||f||_{\ell^\infty} = c$. Fix $m \in \mathbb{Z}$ with $m > n$ (the case $m < n$ is similar). Given $\varepsilon > 0$, there exists $r, s > 0$ sufficiently large such that

$$\frac{1}{r+s+1} \left\{ \sum_{k=-r}^{k=s} f(n+k) \right\} \geq \widetilde{M}f(n) - \varepsilon.$$

Now, consider the average of length $r+s+1$ for m:

$$\begin{aligned}
\widetilde{M}f(m) &\geq \frac{1}{r+s+1}\left\{\sum_{k=-r}^{k=s} f(m+k)\right\} \\
&= \frac{1}{r+s+1}\left\{\sum_{k=-r}^{k=s} f(n+k)\right\} \\
&+ \frac{1}{r+s+1}\left\{\sum_{k=m-n-r}^{k=m-n+s} f(n+k) - \sum_{k=-r}^{k=s} f(n+k)\right\} \\
&\geq \left(\widetilde{M}f(n) - \varepsilon\right) - \frac{2c|m-n|}{r+s+1}.
\end{aligned}$$

□

Now we are ready to prove the first theorem of this chapter.

Proof of Theorem 4.1. Notice that the equality is attained in Example 4.1, this proves the the constant $C = 1$ is the best possible for an inequality of the form $\operatorname{Var}(\widetilde{M}f) \leq C \operatorname{Var}(f)$. If f has bounded variation, we have

$$|f(n)| \leq |f(0)| + |f(n) - f(0)| \leq |f(0)| + \operatorname{Var}(f) < \infty$$

and therefore f is bounded. Without loss of generality, we can assume that f is non-negative since $\operatorname{Var}(|f|) \leq \operatorname{Var}(f)$. Observe that to study the variation of $\widetilde{M}f$, we can consider only its values on the critical points. Even more, we can reduce the analysis of it to and alternating sequence of local maxima $\{a_i\}_{i\in\mathbb{Z}}$ and local minima $\{b_i\}_{i\in\mathbb{Z}}$,

$$\cdots < b_{-2} < a_{-2} < b_{-1} < a_{-1} < b_0 < a_0 < b_1 < a_1 < b_2 < a_2 < \ldots \tag{4.1}$$

We choose this sequence in a way that the second equality in (4.2) holds. The sequence (4.1) can be infinite or finite depending the behavior of $\widetilde{M}f$ asymptotically. Therefore, consider the following two cases.

Case 1. If the sequence (4.1) is infinite in both sides, using the Lemma 4.4 we have

$$\begin{aligned}\mathrm{Var}(\widetilde{M}f) &= \sum_{n=-\infty}^{n=\infty} |\widetilde{M}f(n+1) - \widetilde{M}f(n)| \\ &= \sum_{i=-\infty}^{i=\infty} (\widetilde{M}f(a_{i-1}) - \widetilde{M}f(b_i)) + (\widetilde{M}f(a_i) - \widetilde{M}f(b_i)) \\ &= \sum_{i=-\infty}^{i=\infty} (f(a_{i-1}) - \widetilde{M}f(b_i)) + (f(a_i) - \widetilde{M}f(b_i)) \\ &\leq \sum_{i=-\infty}^{i=\infty} (f(a_{i-1}) - f(b_i)) + (f(a_i) - f(b_i)) \\ &\leq \mathrm{Var}(f).\end{aligned} \tag{4.2}$$

Case 2. If the sequence (4.1) is finite in one or both sides, the cases can be treated in the same way using (4.2) with few modifications. Observe that if last critical point a_k is a local maximum, the sequence $\widetilde{M}f(n)$ becomes non-increasing, and since it is bounded, the limit of the tail exists, say

$$\lim_{n\to\infty} \widetilde{M}f(n) = c$$

and then

$$\liminf_{n\to\infty} f(n) \leq c.$$

Denote the local variation as

$$\mathrm{Var}(f)_{[a,b]} = \sum_{n=a}^{b-1} |f(n+1) - f(n)|$$

for $a, b \in \mathbb{Z}$, possibly $\pm\infty$. Therefore using Lemma 4.4 we have,

$$\begin{aligned}\mathrm{Var}(\widetilde{M}f) &= \mathrm{Var}(\widetilde{M}f)_{[-\infty,a_k]} + \mathrm{Var}(\widetilde{M}f)_{[a_k,+\infty]} \\ &= \sum_{i=-\infty}^{i=k} (\widetilde{M}f(a_{i-1}) - \widetilde{M}f(b_i)) + (\widetilde{M}f(a_i) - \widetilde{M}f(b_i)) + (\widetilde{M}f(a_k) - c) \\ &= \sum_{i=-\infty}^{i=k} (f(a_{i-1}) - f(b_i)) + (f(a_i) - f(b_i)) + (f(a_k) - c) \\ &\leq \mathrm{Var}(f)_{[-\infty,a_k]} + \mathrm{Var}(f)_{[a_k,+\infty]} = \mathrm{Var}(f).\end{aligned}$$

The other cases are treated with minor differences. □

4.1.2 On the centered discrete maximal function

The authors in [1, 8] have studied an analogous result for Theorem 4.1, this is not an easy task, however, they considered $f \in \ell^1$ which is an stronger condition.

Proof of Theorem 4.2. Using Example 4.1 the equality is attained. By Proposition 4.3, we know that for all $n \in \mathbb{Z}$ there is an $r_n \in \mathbb{Z}^+$, such that

$$Mf(n) = A_{r_n} f(n) := \frac{1}{2r_n + 1} \sum_{k=-r_n}^{k=r_n} |f(n+k)|.$$

Define the sets

$$X^- = \{n \in \mathbb{Z}; Mf(n) \geq Mf(n+1)\}$$

and

$$X^+ = \{n \in \mathbb{Z}; Mf(n) < Mf(n+1)\}.$$

Consequently

$$\begin{aligned} \mathrm{Var}(Mf) &= \sum_{n \in X^-} Mf(n) - Mf(n+1) + \sum_{n \in X^+} Mf(n+1) - Mf(n) \\ &\leq \sum_{n \in X^-} A_{r_n} f(n) - A_{r_n+1} f(n+1) \\ &\quad + \sum_{n \in X^+} A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n). \end{aligned} \tag{4.3}$$

The idea of the proof consist in estimating the contribution to (4.3) of the term corresponding to $f(m)$ for any given $m \in \mathbb{Z}$. Without lost of generality, we can assume that f is non-negative.

Case 1. If $n \in X^-$ and $n \geq m$. In this case, the term corresponding to $f(m)$ in $A_{r_n} f(n) - A_{r_n+1} f(n+1)$ is 0 if $m < n - r_n$ or $\frac{f(m)}{2r_n+1} - \frac{f(m)}{2r_n+3}$ if $n - r_n \leq m$, in the last case we have

$$\begin{aligned} \frac{f(m)}{2r_n + 1} - \frac{f(m)}{2r_n + 3} &= \frac{2f(m)}{(2r_n + 1)(2r_n + 3)} \\ &\leq \frac{2f(m)}{(2(n-m) + 1)(2(n-m) + 3)} \\ &= \frac{f(m)}{2(n-m) + 1} - \frac{f(m)}{2(n-m) + 3}. \end{aligned}$$

Case 2. If $n \in X^+$ and $n \geq m$. In this case, the term corresponding to $f(m)$ in $A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)$ is non-positive if $m < n - r_{n+1} + 1$ or $\frac{f(m)}{2r_{n+1}+1} - \frac{f(m)}{2r_{n+1}+3}$ if $m \geq n - r_{n+1} + 1$, in the last case we have

$$\begin{aligned} \frac{f(m)}{2r_{n+1}+1} - \frac{f(m)}{2r_{n+1}+3} &= \frac{2f(m)}{(2r_{n+1}+1)(2r_{n+1}+3)} \\ &\leq \frac{2f(m)}{(2(n-m+1)+1)(2(n-m+1)+3)} \\ &= \frac{f(m)}{2(n-m+1)+1} - \frac{f(m)}{2(n-m+1)+3} \\ &< \frac{f(m)}{2(n-m)+1} - \frac{f(m)}{2(n-m)+3}. \end{aligned}$$

Case 3. If $n \in X^-$ and $n < m$. In this case, the term corresponding to $f(m)$ in $A_{r_n} f(n) - A_{r_n+1} f(n+1)$ is non-positive if $m > n + r_n$ or $\frac{f(m)}{2r_n+1} - \frac{f(m)}{2r_n+3}$ if $m \leq n + r_n$, in the last case we have

$$\begin{aligned} \frac{f(m)}{2r_n+1} - \frac{f(m)}{2r_n+3} &= \frac{2f(m)}{(2r_n+1)(2r_n+3)} \\ &\leq \frac{2f(m)}{(2(m-n)+1)(2(m-n)+3)} \\ &= \frac{f(m)}{2(m-n)+1} - \frac{f(m)}{2(m-n)+3} \\ &< \frac{f(m)}{2(m-n-1)+1} - \frac{f(m)}{2(m-n-1)+3}. \end{aligned}$$

Case 4. If $n \in X^+$ and $n < m$. In this case, the term corresponding to $f(m)$ in $A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)$ is 0 if $m > n + r_{n+1} + 1$ or $\frac{f(m)}{2r_{n+1}+1} - \frac{f(m)}{2r_{n+1}+3}$ if $m \leq n + r_{n+1} + 1$, in the last case we have

$$\begin{aligned} \frac{f(m)}{2r_{n+1}+1} - \frac{f(m)}{2r_{n+1}+3} &= \frac{2f(m)}{(2r_{n+1}+1)(2r_{n+1}+3)} \\ &\leq \frac{2f(m)}{(2(m-n-1)+1)(2(m-n-1)+3)} \\ &= \frac{f(m)}{2(m-n-1)+1} - \frac{f(m)}{2(m-n-1)+3}. \end{aligned}$$

This implies that the terms corresponding to $f(m)$ in (4.3) are bounded by the expression

$$\sum_{n\geq m} \frac{f(m)}{2(n-m)+1} - \frac{f(m)}{2(n-m)+3} \\ + \sum_{n<m} \frac{f(m)}{2(m-n-1)+1} - \frac{f(m)}{2(m-n-1)+3} = 2f(m).$$

In conclusion, we have proved the theorem since

$$\mathrm{Var}(Mf) \leq \sum_{m=-\infty}^{m=\infty} 2f(m) = 2||f||_{\ell^1(\mathbb{Z})}.$$

$\square$

Chapter 5

Maximal operators of convolution type

In this chapter, we will introduce two maximal operators, the heat maximal operator and the Poisson maximal operator. The reason for their names will appear soon after their definitions. This part of the report is based on the paper [4].

Our purpose is to present some results on these maximal operators defined by convolution. In the paper [4], the authors address the question of whether these maximal operators can decrease the variation (or L^p-variation) of a function, also if this variation is bounded with respect to that of the initial data. In other words, the results in this paper shows under what hypothesis we can prove that:

- For $p \geq 1$ and a function $f \in W^{1,p}(\mathbb{R}^d)$, the maximal function of f, denoted by $M_\varphi f$ belongs to $W^{1,p}(\mathbb{R}^d)$ and the inequality

  $$||DM_\varphi f||_p \leq C||Df||_p$$

  holds for $C = 1$.

- For $f \in BV(\mathbb{R})$, then $M_\varphi f \in BV(\mathbb{R})$ and the inequality

  $$\mathrm{Var}(M_\varphi f) \leq C\,\mathrm{Var}(f)$$

  holds for $C = 1$.

To clarify, given a function $\varphi \in L^1(\mathbb{R}^d)$, such that $\int_{\mathbb{R}^d} \varphi = 1$, we let the approximation of identity to be $\varphi_t(x) = t^{-d}\varphi(x/t)$. The maximal operator $M_\varphi f$ is defined as

$$M_\varphi f(x) = \sup_{t>0}(|f| * \varphi_t)(x).$$

Note that this definition coincides with the Hardy-Littlewood maximal function when we use the indicator function $\varphi(x) = \chi_{B(0,1)}/m(B(0,1))$.

An important comment is that the Theorem 2, Chapter III in [10] give us the pointwise inequality

$$M_{\varphi} f(x) \leq A\, Mf(x) \tag{5.1}$$

for $A > 0$. This implies that the operator M_{φ} is of weak type $(1,1)$ and strong type (p,p) for $1 < p \leq \infty$. Now, repeating the arguments in Theorem 2.1, we can prove that for $p > 1$,

$$||DM_{\varphi} f||_p \leq C\, ||Df||_p \tag{5.2}$$

for a constant $C > 1$. Hence, using the inequalities (5.1) and (5.2), $M_{\varphi} : W^{1,p}(\mathbb{R}^d) \to W^{1,p}(\mathbb{R}^d)$ is a bounded operator.

For the sake of our discussion, first we give some definitions and announce the theorems of [4] and finally we give their proofs.

5.1 Poisson maximal operator

Given $f_0 \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$ and the Poisson kernel given by

$$P_y(x) = \frac{c_d\, y}{(|x|^2 + y^2)^{\frac{d+1}{2}}} \text{ with } c_d = \frac{\Gamma\left(\frac{d+1}{2}\right)}{\pi^{\frac{d+1}{2}}},$$

the **Poisson maximal operator** is given by

$$f^{\star}(x) = \sup_{y>0} (|f_0| * P_y)(x). \tag{5.3}$$

As in the heat maximal operator, the Poisson maximal operator is motivated by the solution to the Laplace's equation

$$\Delta f = 0 \text{ in } \mathbb{R} \times (0, \infty),$$

with boundary condition

$$\lim_{y \to 0^+} f(x,y) = f_0(x).$$

The solution to this problem is given by

$$f(x,y) = (f_0 * P_y)(x). \tag{5.4}$$

We prove the following theorem, which is analogous to the Theorem 5.2.

Theorem 5.1. *Let $f^\star$ be the Poisson maximal function in* (5.3). *The following propositions are true.*

(i) Let $1 < p \leq \infty$ and $f_0 \in W^{1,p}(\mathbb{R})$. Then $f^\star \in W^{1,p}(\mathbb{R})$ and

$$||(f^\star)'||_p \leq ||f_0'||_p.$$

(ii) Let $f_0 \in W^{1,1}(\mathbb{R})$. Then $f^\star \in L^\infty(\mathbb{R})$ and has a weak derivative $(f^\star)'$ that satisfies

$$||(f^\star)'||_1 \leq ||f_0'||_1.$$

(iii) Let $f_0 : \mathbb{R} \to \mathbb{R}$ be a function of bounded variation. Then $f^\star$ is a function of bounded variation and

$$Var(f^\star) \leq Var(f_0).$$

(iv) Let $d > 1$ and $f_0 \in W^{1,p}(\mathbb{R}^d)$, with $p = 2$ or $p = \infty$. Then $f^\star \in W^{1,p}(\mathbb{R}^d)$ and

$$||D(f^\star)||_p \leq ||Df_0||_p.$$

5.2 Heat flow maximal operator

Given $f_0 \in L^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$ and the Gauss kernel or heat kernel given by

$$K_t(x) = \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/t},$$

the **heat flow maximal operator** is given by

$$f^*(x) = \sup_{t>0}(|f_0| * K_t)(x). \tag{5.5}$$

An important motivation to consider this maximal operator is that the function $f : \mathbb{R}^d \times (0, \infty) \to \mathbb{R}$ defined as

$$f(x,t) = (f_0 * K_t)(x)$$

solves the heat equation

$$\partial_t f + \Delta_x f = 0, \text{ in } \mathbb{R}^d \times (0, \infty)$$

with boundary condition

$$\lim_{t\to 0*} f(x,t) = f_0(x), \text{ almost everywhere } x \in \mathbb{R}^d.$$

We prove the following theorem about this maximal operator.

Theorem 5.2. *Let f^* be the heat flow maximal function in* (5.5). *The following propositions are true.*

(i) Let $1 < p \leq \infty$ and $f_0 \in W^{1,p}(\mathbb{R})$. Then $f^* \in W^{1,p}(\mathbb{R})$ and

$$||(f^*)'||_p \leq ||f_0'||_p.$$

(ii) Let $f_0 \in W^{1,1}(\mathbb{R})$. Then $f^* \in L^\infty$ and has a weak derivative $(f^*)'$ that satisfies

$$||(f^*)'||_1 \leq ||f_0'||_1.$$

(iii) Let $f_0 \in \mathrm{BV}(\mathbb{R})$ be a function of bounded variation. Then $f^* \in \mathrm{BV}(\mathbb{R})$ and

$$Var(f^*) \leq Var(f_0).$$

(iv) Let $d > 1$ and $f_0 \in W^{1,p}(\mathbb{R}^d)$, with $p = 2$ or $p = \infty$. Then $f^* \in W^{1,p}(\mathbb{R}^d)$ and

$$||D(f^*)||_p \leq ||Df_0||_p.$$

5.3 The regularity of Poisson maximal function: Proof of Theorem 5.1

Our aim is to prove the Theorem 5.1, for simplicity, we first assume that the initial condition f_0 is non negative. On one hand, if $f_0 \in W^{1,p}(\mathbb{R}^d)$, from (2.4), we know that $|f_0| \in W^{1,p}(\mathbb{R}^d)$, and $||D|f_0|||_p = ||Df_0||_p$ for $d \geq 1$ and $1 \leq p \leq \infty$. On the other hand, it is well known that $\mathrm{Var}(|f_0|) \leq \mathrm{Var}(f_0)$ if f_0 is a function of bounded variation.

Also, it would be useful to have in mind some properties of the Poisson kernel, see for instance section 2 chapter III in [10].

Lemma 5.3. *The following statements are true*

(i) If $f_0 \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$, then $f^\star \in C(\mathbb{R}^d)$.

(ii) If f_0 is bounded and Lipschitz continuous, then $f^\star$ is bounded and Lipschitz continuous with $\mathrm{Lip}(f^\star) \leq \mathrm{Lip}(f_0)$.

Proof. *(i)* Let $\varepsilon > 0$. Given any $h \in \mathbb{R}^d$ and some $0 < y_\varepsilon < \infty$, we have

$$
\begin{aligned}
|(\tau_h f_0 - f_0) * P_y(x)| &\le \left(|\tau_h f_0 - f_0|^p * P_y^p(x)\right)^{1/p} \\
&\le \left(||\tau_h f_0 - f_0||_p^p \, ||P_y||_\infty^p\right)^{1/p} \\
&= ||\tau_h f_0 - f_0||_p \, ||P_y||_\infty \\
&\le \frac{c_d ||\tau_h f_0 - f_0||_p}{y^d} \\
&< \varepsilon
\end{aligned}
\tag{5.6}
$$

if $y \ge y_\varepsilon$. We have used Jensen's inequality in the first line and Young's inequality in the second, see Theorem 4.15 in [2]. When $y \le y_\varepsilon$, we can split the convolution in two integrals and using p' as the conjugate exponent of p we have

$$
\begin{aligned}
&|(\tau_h f_0 - f_0) * P_y(x)| \le |(\tau_h f_0 - f_0)| * P_y(x) \\
&\quad = \int_{|z| \le \frac{y_\varepsilon}{2}} |(\tau_h f_0 - f_0)(z - x)| P_y(z) dz + \int_{|z| \ge \frac{y_\varepsilon}{2}} |(\tau_h f_0 - f_0)(z - x)| P_y(z) dz \\
&\quad \le \max_{z \in B(x, \frac{y_\varepsilon}{2})} |(\tau_h f_0 - f_0)(z)| \int_{|z| \le \frac{y_\varepsilon}{2}} P_y(z) dz + ||\tau_h f_0 - f_0||_p ||\chi_{\{|x| \ge \frac{y_\varepsilon}{2}\}} P_y||_{p'} \\
&\quad \le \max_{|z| \le \frac{y_\varepsilon}{2}} |(\tau_h f_0 - f_0)(z)| + ||\tau_h f_0 - f_0||_p ||\chi_{\left\{|x| \ge \frac{y_\varepsilon}{2}\right\}} P_y||_{p'}.
\end{aligned}
$$

From the inequality above, we can say the following

- The norm $\left|\left|\chi_{\left\{|x| \ge \frac{y_\varepsilon}{2}\right\}} P_y\right|\right|_{p'}$ is bounded for $0 < y < y_\varepsilon$. If $p' < \infty$, using polar coordinates

$$
\begin{aligned}
\left|\left|\chi_{\left\{|x| \ge \frac{y_\varepsilon}{2}\right\}} P_y\right|\right|_{p'}^{p'} &= \int_{\frac{y_\varepsilon}{2}}^{\infty} \left(c_d \frac{y}{(r^2 + y^2)^{\frac{d+1}{2}}}\right)^{p'} r^{d-1} dr \\
&\le (c_d y_\varepsilon)^{p'} \int_{\frac{y_\varepsilon}{2}}^{\infty} r^{(d-1) - p'(d+1)} dr \\
&< \infty.
\end{aligned}
$$

  The last inequality follows from the fact $1 < p'$ and this implies that the exponent satisfies $d - p'(d+1) < -1$. Also, if $p' = \infty$, being $P_y(x)$ decreasing on $|x|$, we have $\left|\left|\chi_{\left\{|x| \ge \frac{y_\varepsilon}{2}\right\}} P_y\right|\right|_\infty \le \frac{c_d 2^{d+1}}{y_\varepsilon^d}$.

- The norm $||\tau_h f_0 - f_0||_p$ goes to zero as $|h| \to 0$, by approximation with a function in $C_c^\infty(\mathbb{R}^d)$ and Corollary 4.23 in [2].

- The value $\max_{|z|\leq \frac{y_\varepsilon}{2}}$ goes to zero as $|h| \to 0$.

Summarizing, there exists $\delta > 0$ such that, if $|h| < \delta$, then

$$|(\tau_h f_0 - f_0) * P_y(x)| < \varepsilon. \tag{5.7}$$

Using the sublinearity of the Poisson maximal operator, the inequalities (5.6) and (5.6), we can conclude that

$$|\tau_h f^\star(x) - f^\star(x)| \leq \varepsilon, \tag{5.8}$$

for $|h| \leq \delta$, in other words, $f^\star \in C(\mathbb{R}^d)$.

(ii) If f_0 is bounded, say by M, the convolution $f_0 * P_y$ is also bounded by M and so the pointwise supremum is also bounded. Regarding to Lipschitz continuity, the idea is the same. If f_0 has Lipschitz constant L, the convolution is also Lipschitz conitinuous and together with sublinearity of the Poisson maximal operator, we can conclude that $f^\star$ is also Lipschitz continuos with Lipschitz constant at most L. □

As in previous chapters, we move to the analysis of the open set where the Poisson maximal function is above the initial data. In this set, we see that the subharmonicity of $f^\star$ is a property inherited from the Poisson kernel. Therefore, we will introduce a new definition. We say that a continuous function f is **subharmonic** in an open set A, if for every $x \in A$ and $r > 0$ such that $\overline{B(x,r)} \subset A$ we have

$$f(x) \leq \frac{1}{r^{d-1}\sigma_{d-1}} \int_{\partial B(x,r)} f(y) d\mathcal{H}^{d-1}(y). \tag{5.9}$$

We use σ_{d-1} to denote the surface area of the unit ball and $\mathcal{H}^{d-1}$ to denote the Hausdorff measure of dimension $d-1$. The expression $\overline{B(x,r)}$ represents the topological closure of the ball with center x and radius r, while $\partial B(x,r)$ represents its boundary.

Lemma 5.4 (Subharmonicity property)**.** *Let $f_0 \in C(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$ for some $1 \leq p < \infty$ or f_0 be bounded and Lipschitz continuous. Then $f^\star$ is subharmonic in the open set $A = \{x \in \mathbb{R}^d; f^\star(x) > f_0(x)\}$.*

Proof. The set A is open since the Lemma 5.3 says that $f^\star$ is continuous, so $f^\star - f_0$ is a continuous function. We will denote $B((x,y),r)$ for an open ball in $\mathbb{R}^{d+1}$ whose center is (x,y) with $x \in \mathbb{R}^d$, $y \in \mathbb{R}$ and radius $r > 0$.

If A is an empty set, there is nothing to prove. Therefore, fix $x_0 \in A$. Remember that $f(x,y)$ is the solution to the Laplace's equation with initial condition f_0 defined in (5.4), we have that $f_0(x_0) < f^\star(x_0)$, and $\lim_{y\to 0^+} f(x_0, y) = f_0(x_0)$. Next, let $\delta > 0$ such that for every $y < \delta$ then

$$f(x_0, y) < f^\star(x_0) - \frac{1}{2}(f^\star(x_0) - f_0(x_0)). \tag{5.10}$$

Let $y_0 \geq \delta$. Choose a radius $0 < r_0 < \delta$ such that $\overline{B(x_0, r_0)} \subset A$. For any $r < r_0$ we have that $\overline{B((x_0, y_0), r)} \subset A \times (0, \infty)$, because it is true that $B((x_0, y_0), r) \subset B(x_0, r) \times (y_0 - r, y_0 + r)$ and $y_0 - r > y_0 - r_0 > 0$. Since $P_y(x)$ is harmonic for $(x, y) \in \mathbb{R}^d \times (0, \infty)$, $f(x, y)$ is also harmonic there, and by the mean value average property

$$f(x_0, y_0) = \frac{1}{r^{d+1}\omega_{d+1}} \int_{B((x_0,y_0),r)} f(x, y)\, dx\, dy \tag{5.11}$$

with ω_{d+1} denoting the volume of the unit sphere of dimension $d + 1$. From the average (5.11) we have

$$\begin{aligned} f(x_0, y_0) &\leq \frac{1}{r^{d+1}\omega_{d+1}} \int_{B((x_0,y_0),r)} f^\star(x)\, dx\, dy \\ &= \frac{1}{r^{d+1}\omega_{d+1}} \int_{B(x_0,r)} 2\sqrt{r^2 - |x - x_0|^2} f^\star(x)\, dx. \end{aligned} \tag{5.12}$$

The estimation of the Poisson solution we have in inequality (5.10) implies that $\sup_{0<y<\delta} f(x_0, y) < f^\star(x_0)$. This says that for every $x_0 \in A$,

$$f^\star(x_0) = \sup_{y\geq\delta} f(x_0, y). \tag{5.13}$$

Since we have obtained the inequality (5.12) for arbitrary $y_0 \geq \delta$, together with (5.13) we have that for every $r < r_0$

$$f^\star(x_0) \leq \frac{1}{r^{d+1}\omega_{d+1}} \int_{B(x_0,r)} 2\sqrt{r^2 - |x - x_0|^2} f^\star(x)\, dx. \tag{5.14}$$

The next step is to show that the average (5.14) implies the subharmonicity of $f^\star$ in A. First, we show that this implies the maximum principle on each connected component of A. As the standard proof, let Ω a connected component of A such that $\overline{\Omega} \subset A$. Denote $M = \sup_\Omega f^\star$ and define $B_1 = \{x \in \Omega; f^\star < M\}$ and $B_2 = \{x \in \Omega; f^\star = M\}$. We prove B_2 is open. Take $z_0 \in B_2$, since $f^\star$ is continuous, take r sufficiently small such that $\overline{B(z_0, r)} \subset \Omega$

$$M = f(z_0) \leq \frac{1}{r^{d+1}\omega_{d+1}} \int_{B(z_0,r)} 2\sqrt{r^2 - |x - x_0|^2} f^\star(x)\, dx \leq M$$

This is enough to show that $B(z_0,r) \subset B_2$ and then $B_2 = \Omega$.

Now, let $x_0 \in A$ and $s > 0$ such that $\overline{B(x_0,s)} \subset A$ and let $h : \overline{B(x_0,s)} \to \mathbb{R}$ the solution of the Dirichlet problem

$$\begin{cases} \Delta h = 0 & \text{on } B(x_0,s), \\ h = f^\star & \text{in } \partial B(x_0,s). \end{cases}$$

Denote by g the difference $f^\star - h$; now we prove it satisfies the same type of average that in (5.14), but on the ball $B(x_0,s)$. Observe that for z_0 and $r > 0$ such that $B(z_0,r) \subset B(x_0,s)$

$$\begin{aligned} g(z_0) &= f^\star(z_0) - h(z_0) \\ &\le \frac{1}{r^{d+1}\omega_{d+1}} \int_{B(z_0,r)} 2\sqrt{r^2 - |x-x_0|^2} f^\star(x)\, dx - h(z_0) \\ &= \frac{1}{r^{d+1}\omega_{d+1}} \int_{B(z_0,r)} 2\sqrt{r^2 - |x-x_0|^2}\, (f^\star(x) - h(x))\, dx \\ &= \frac{1}{r^{d+1}\omega_{d+1}} \int_{B(z_0,r)} 2\sqrt{r^2 - |x-x_0|^2} g(x)\, dx. \end{aligned} \tag{5.15}$$

The third line is true since $h(x)$ is also harmonic in the ball $B((z_0,0),r)$ and it satisfies the mean average formula. In addition, the inequality (5.15) implies the maximum principle in $B(x_0,s)$. Therefore, the maximum value of g must be attained in $\partial B(x_0,s)$, but $g = 0$ in this boundary, hence $f^\star(x_0) - h(x_0) \le 0$. Using the harmonicity of h and that it equals $f^\star$ in $\partial B(x_0,s)$ we have that

$$\begin{aligned} f^\star(x_0) \le h(x_0) &= \frac{1}{r^{d-1}\sigma_{d-1}} \int_{\partial B(x_0,s)} h(y) d\mathcal{H}^{d-1}(y) \\ &= \frac{1}{r^{d-1}\sigma_{d-1}} \int_{\partial B(x_0,s)} f^\star(y) d\mathcal{H}^{d-1}(y). \end{aligned} \tag{5.16}$$

In conclusion, the inequality (5.16) means that $f^\star$ is subharmonic in A. $\square$

The following lemma is an important reduction of the assumptions we can make, it is based on the fact that the Poisson kernel satisfies the semigroup property $P_{y_1} * P_{y_2} = P_{y_1+y_2}$. This is verified using the Fourier transform $\widehat{P_y}(x) = e^{-2\pi|x|y}$ and the Fourier inversion formula of Proposition 5 Chapter III in [10].

Lemma 5.5 (Reduction to Lipschitz continuity)**.** *We can assume without loss of generality, that f_0 is Lipschitz continuous in part (i) and (iv) of Theorem 5.1.*

Proof. If $p = \infty$, there is a relation between Lipschitz functions and Sobolev functions. It is true that $f_0 \in W^{1,p}(\mathbb{R}^d)$ if and only if f_0 has a representative that is bounded and Lipschitz continuous.

If $1 < p < \infty$, we take $\varepsilon > 0$ and define

$$f_\varepsilon(x) := f_0 * P_\varepsilon(x).$$

We observe that f_ε is Lispchitz continuous since is differentiable and the derivative is bounded using the Young's inequality. Now assume the statements of the theorem are true for f_ε, which means, for $f_\varepsilon^\star(x) = \sup_{y>0} f_\varepsilon * P_y(x) = \sup_{y>\varepsilon} f_0 * P_y(x)$, it is true that $f_\varepsilon^\star \in W^{1,p}((R)^d)$ and

$$||Df_\varepsilon^\star||_p \leq ||Df_\varepsilon||_p. \tag{5.17}$$

On one hand, the Young's inequality implies that $||f_\epsilon||_p \leq ||f_0||_p ||P_y||_1 = ||f_0||_p$. On the other hand, the Minkowski's inequality implies that for every $1 \leq i \leq d$,

$$\begin{aligned}
\int_{\mathbb{R}^d} (D_i f_\varepsilon(x))^p \, dx &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} (D_i f_0)(x-z) P_\varepsilon(z) \, dz \right)^p dx \\
&\leq \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \{(D_i f_0)(x-z) P_\varepsilon(z)\}^p \, dx \right)^{1/p} dz \right)^p \\
&= ||D_i f_0||_p^p,
\end{aligned}$$

which implies that $||Df_\varepsilon||_p \leq ||Df_0||_p$. The inequality (5.17) and the observations above prove that the sequence $\{Df_\varepsilon^\star\}$ is bounded in $L^p(\mathbb{R}^d)$ by Df_0. Also, we have that $f_\varepsilon^\star$ converges pointwise to $f^\star$ as $\varepsilon \to 0$. This is enough to argue that $\{f_\varepsilon^\star\}$ is a bounded sequence in $W^{1,p}(\mathbb{R}^d)$. Together with the weak compactness of $W^{1,p}(\mathbb{R}^d)$, as in the proof of the Theorem 2.2, we can conclude that $f^\star \in W^{1,p}(\mathbb{R}^d)$. Finally, by Fatou's lemma we can prove that

$$||Df^\star||_p \leq \liminf_{\varepsilon \to 0} ||Df^\star||_p \leq \liminf_{\varepsilon \to 0} ||Df_\varepsilon|| \leq ||Df_0||_p.$$

□

The following lemma is useful to prove the part (iv).

Lemma 5.6. *Let f, $g \in C(\mathbb{R}^d) \cap W^{1,2}(\mathbb{R}^d)$ with g Lipschitz continuous and nonnegative, let f be subharmonic on the open set $J = \{x \in \mathbb{R}^d; g(x) > 0\}$. Then*

$$\int_{\mathbb{R}^d} \langle Df, Dg \rangle \, dx \leq 0$$

Proof. We will prove the result for g with compact support. The reason to this simplification is as follows. Let $\psi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative function such that $0 \le \psi(x) \le 1$, with support contained in the ball $B(0,2)$ and $\psi(x) = 1$ for every $x \in B(0,1)$. We denote $\psi_n(x) := \psi(x/n)$ and the functions with compact support $g_n(x) := g(x)\psi_n(x)$. We have that $g_n \to g$ in $W^{1,2}(\mathbb{R}^d)$ since the convergence $g_n(x) \to g(x)$ and $D\psi_n(x) = n^{-1}D\psi(n^{-1}x) \to 0$ are pointwise and Dg_n is uniformly bounded because g is bounded and Lipschitz continuous. Also, we have the set equality

$$J = \cup_{n=1}^\infty \{x \in \mathbb{R}^d; g_n(x) > 0\}.$$

Summarizing, each g_n is Lipschitz continuous and f is subharmonic on the sets $\{x \in \mathbb{R}^d; g_n(x) > 0\}$, hence

$$\int_{\mathbb{R}^d} \langle Df, Dg\rangle \, dx = \lim_{n\to\infty} \int_{\mathbb{R}^d} \langle Df, Dg_n\rangle \, dx \le 0.$$

The equality above follows from the fact that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \langle Df, Dg - Dg_n\rangle \, dx \le ||Df||_2 \lim_{n\to\infty} ||Dg - Dg_n||_2 = 0.$$

Let $\phi \in C_c^\infty(\mathbb{R}^d)$ be a nonnegative function with support on $B(0,1)$ and $\int_{\mathbb{R}^d} \phi \, dx = 1$. For $\varepsilon > 0$, define the approximation of identity $\phi_\varepsilon(x) := \varepsilon^{-d}\phi(\varepsilon^{-1}x)$ and let f_ε be the convolution $f * \phi_\varepsilon$. We see that f_ε is subharmonic on the set $J_\varepsilon = \{x \in \mathbb{R}^d; \mathrm{dist}(x, \partial J)\}$. For $x \in J_\varepsilon$ and $r > 0$ such that $\overline{B(x,r)} \subset J_\varepsilon$, we have

$$\begin{aligned} f_\varepsilon(x) &= \int_{\mathbb{R}^d} f(x-y)\phi_\varepsilon(y)\, dy \\ &\le \int_{\mathbb{R}^d} \frac{1}{r^{d-1}\sigma_{d-1}} \int_{\partial B(0,r)} f(x-y+z) d\mathcal{H}^{d-1}(z)\phi_\varepsilon(y)\, dy \\ &= \frac{1}{r^{d-1}\sigma_{d-1}} \int_{\partial B(0,r)} f_\varepsilon(x+z)\, d\mathcal{H}^{d-1}(z). \end{aligned} \tag{5.18}$$

In addition, the Laplacian Δf_ε is nonnegative on J_ε since $f_\varepsilon \in C^\infty(\mathbb{R}^d)$ and satisfies (5.18).

By standard argument of approximations to g with smooth functions of compact support, we can prove that the formula of integration by parts holds, this gives us

$$\begin{aligned} \int_{\mathbb{R}^d} \langle Df_\varepsilon, Dg\rangle \, dx &= \int_{\mathbb{R}^d} (-\Delta f_\varepsilon) g \, dx \\ &\le \int_{J\setminus J_\varepsilon} (-\Delta f_\varepsilon) g \, dx \end{aligned} \tag{5.19}$$

due to the product $(-\Delta f_\varepsilon)g$ is nonpositive on J_ε.

Let $x \in J\backslash J_\varepsilon$ and $y \in \partial J$, since g is Lipschitz continuous,

$$|g(x)| = |g(x) - g(y)| \leq \text{Lip}(g)|x - y| \leq \text{Lip}(g)\varepsilon.$$

This implies that

$$\begin{aligned}
\int_{J\backslash J_\varepsilon} |(-\Delta f_\varepsilon)g|\, dx &\leq \text{Lip}(g)\varepsilon \int_{J\backslash J_\varepsilon} |\Delta f_\varepsilon|\, dx \\
&= \text{Lip}(g)\, \varepsilon \int_{J\backslash J_\varepsilon} \sum_{i=1}^{d} |D_i f * \varepsilon^{-1}(D_i\phi)_\varepsilon)|\, dx \\
&\leq \text{Lip}(g) \int_{J\backslash J_\varepsilon} |Df| * (|D\phi|)_\varepsilon\, dx \\
&\leq \text{Lip}(g)\, |||Df| * (|D\phi|)_\varepsilon||_2 (m(J\backslash J_\varepsilon))^{1/2} \\
&\leq \text{Lip}(g)\, ||Df||_2 ||D\phi||_1 (m(J\backslash J_\varepsilon))^{1/2}
\end{aligned} \tag{5.20}$$

using Cauchy-Schwarz inequality to get rid of the summation and then again to reduce our expression to L^2-norm and finally Young's inequality. From inequalities (5.19) and (5.20), we finally prove that

$$\begin{aligned}
\int_{\mathbb{R}^d} \langle Df, Dg\rangle\, dx &= \lim_{\varepsilon\to 0} \int_{\mathbb{R}^d} \langle Df_\varepsilon, Dg\rangle\, dx \\
&\leq \lim_{\varepsilon\to 0} C(m(J\backslash J_\varepsilon))^{1/2} = 0
\end{aligned}$$

based on the fact that the set J is bounded. □

5.3.1 Proof of part (iv) - Theorem 5.1

For $p = \infty$, we know that $f_0 \in W^{1,\infty}(\mathbb{R}^d)$ can be considered Lipschitz by the Lemma 5.5. Now, from Lemma 5.3, $f^\star$ is also bounded and Lispchitz continuous, which means that $f^\star \in W^{1,\infty}(\mathbb{R}^d)$ with $||Df^\star||_\infty = \text{Lip}(f^\star) \leq \text{Lip}(f_0) = ||Df_0||_\infty$.

For $p = 2$, using Lemma 5.5 we assume that $f_0 \in W^{1,2}(\mathbb{R}^d)$ is Lipschitz, and therefore $f^\star$ is also Lispchitz by Lemma 5.3. By the comments after the inequalities (5.1) and (5.2), we know that $f^\star \in W^{1,2}(\mathbb{R}^d)$. By the Lemma 5.4, on the set $\{x \in \mathbb{R}^d; f^\star(x) > f_0(x)\}$, $f^\star$ is subharmonic. We let $f = f^\star$ and $g = f^\star - f_0$ in Lemma 5.6, then

$\int_{\mathbb{R}^d} \langle Df^\star, D(f^\star - f_0)\rangle\, dx \leq 0$. Now, the inequality

$$\begin{aligned}
||Df^\star||_2^2 &= \int_{\mathbb{R}^d} |Df^\star|^2\, dx \\
&\leq \int_{\mathbb{R}^d} |Df^\star|^2\, dx - 2\int_{\mathbb{R}^d} \langle Df^\star, D(f^\star - f_0)\rangle\, dx + \int_{\mathbb{R}^d} |D(f^\star - f_0)|^2\, dx \\
&= \int_{\mathbb{R}^d} |Df^\star - D(f^\star - f_0)|^2\, dx \\
&= ||Df_0||_2^2
\end{aligned}$$

implies the result we want.

5.3.2 Proof of part (i) - Theorem 5.1

We can focus on the case $1 < p < \infty$ since the case $p = \infty$ was already proved above without regarding the dimension. As before, we can assume that f_0 is Lipschitz continuous by the Lemma 5.5 and then also $f^\star$ is Lipschitz continuous by the Lemma 5.3 and subharmonic in the open set $A := \{x \in \mathbb{R}; f^\star(x) > f_0(x)\}$ by the Lemma 5.4.

We can write the set A as the countable union of disjoint sets $A = \cup_j I_j = \cup_j(\alpha_j, \beta_j)$, and being $f^\star$ subharmonic on A, it must be a continuous convex function in each (α_j, β_j).

Now we will use the Zorn's lemma to find an important function in $W^{1,p}(\mathbb{R})$. Define the family of functions as

$$\mathcal{S} := \left\{ \begin{array}{l} h : \mathbb{R} \to \mathbb{R}, \\ f_0(x) \leq h(x) \leq f^\star(x) \quad \text{for all } x \in \mathbb{R}, \\ \mathrm{Lip}(h) \leq \mathrm{Lip}(f_0), \\ ||h'||_p \leq ||f_0'||_p. \end{array} \right\}.$$

The family $\mathcal{S}$ is non-empty since it contains f_0. We will add a partial order and prove it is inductive, as consequence of the Zorn's lemma it has a maximal element denoted by g. We will consider that $h_1 \preceq h_2$ if and only if $h_1(x) \leq h_2(x)$ for all $x \in \mathbb{R}$. This partial order makes the family $\mathcal{S}$ inductive, for a totally order subset $\{h_\alpha\}_{\alpha\in\Lambda}$, we can prove that the pointwise supremum

$$\bar{h}(x) = \sup_{\alpha\in\Lambda} h_\alpha(x)$$

is an upper bound and $\bar{h} \in \mathcal{S}$. First of all, since a pointwise supremum of Lipschitz continuous function is Lipschitz continuous, $\bar{h}$ is Lipschitz continuous and $\mathrm{Lip}(\bar{h}) \leq$

$\mathrm{Lip}(f_0)$. For each $N \in \mathbb{N}$, we can consider the partition of $[-N, N]$ given by the $2N^2+1$ points $\{j/N\}$, with $-N^2 \le j \le N^2$. For every element of the partition j/N and for every N, choose $h_{j,N} \in \{h_\alpha\}_{\alpha\in\Lambda}$ such that

$$\bar{h}(j/N) - h_{j,N}(j/N) < 1/N.$$

With these choices we can construct the sequence $\{h_N\}_{N\in\mathbb{N}}$ as

$$h_N(x) = \max_{-N^2\le j\le N^2} h_{j,N}(x).$$

We have the pointwise convergence $h_N \to \bar{h}$. The sequence h_N is pointwise increasing with respect to N, and for every $\varepsilon > 0$ and $x \in \mathbb{R}$, there exists $N \gg 1$ such that $x \in [-N, N]$ and

$$\begin{aligned}
&\bar{h}(x) - h_N(x) \\
&\quad = [\bar{h}(x) - \bar{h}(j/N)] + [\bar{h}(j/N) - h_N(j/N)] + [h_N(j/N) - h_N(x)] \\
&\quad \le \mathrm{Lip}(\bar{h})|x - j/N| + 1/N + \mathrm{Lip}(f_0)|x - j/N| \\
&\quad \le 1/N + \mathrm{Lip}(f_0)/N \\
&\quad < \varepsilon,
\end{aligned} \tag{5.21}$$

where $-N^2 \le j \le N^2$ is chosen to satisfy $|x - j/N| \le 1/(2N)$. By construction we have that the sequence $\{h_N\}$ is bounded in $W^{1,p}(\mathbb{R})$, to be precise $||h_N||_{1,p} \le ||f^\star||_p + ||f_0'||_p < \infty$. The weak compactness of the Sobolev spaces implies that h_N converges weakly to $\bar{h} \in W^{1,p}(\mathbb{R})$. By Fatou's lemma

$$||\bar{h}'||_p \le \liminf_{N\to\infty} ||h_N'||_p \le ||f_0'||_p,$$

which implies that $\bar{h} \in \mathcal{S}$. We conclude the verification of the inductive property, and then we can guarantee the existence of an element $g \in \mathcal{S}$ which is maximal.

The next step is to show that this especial element g coincides with $f^\star$. Suppose by contradiction that the open set $B = \{x \in \mathbb{R}; f^\star(x) > g(x)\} \subset A$ is non-empty. We can write B as a countable union of disjoint intervals $\cup_j J_j = \cup_j(\gamma_j, \delta_j)$.

The point is that g cannot be superharmonic in B. If it is and one of the intervals (γ_j, δ_j) is bounded, the maximal principle implies that $f^\star$ equals g on $[\gamma_j, \delta_j]$; notice that $f^\star(\gamma_j) = g(\gamma_j)$ and $f^\star(\delta_j) = g(\delta_j)$ and the difference $f^\star - g$ is subharmonic in B. The maximal property implies $f^\star - g \le 0$ on (γ_j, δ_j) since the equality is satisfied in the boundary. If an interval J_j is unbounded, say (γ_j, ∞) (similarly $(-\infty, \delta_j)$), the function $f^\star - g$ is strictly positive and convex, but this is not possible. On one hand, $(f^\star - g)(\gamma_j) = 0$. On the other, $f^\star - g$ is Lipschitz continuous and belongs to $L^p(\mathbb{R})$, therefore $\lim_{x\to+\infty}(f^\star - g)(x) = 0$. This implies that $f^\star = g$ on (γ_j, ∞). The case $B = \mathbb{R}$ cannot happen, because f_0 coincide with $f^\star$, in a point x_0, where attains its global maximum. This can be explained by the inequality

$$f_0(x_0) \le g(x_0) \le f^\star(x_0) \le Mf(x_0) = f(x_0).$$

The discussion above guarantees that we can find a closed interval $[a,b] \subset B$ such that

$$g\left(\frac{a+b}{2}\right) < \frac{g(a)+g(b)}{2}.$$

Let $\ell(x)$ the parametrization of the segment that connect the points $(a, g(a))$ and $(b, g(b))$ given by

$$\ell(x) = \frac{g(b)-g(a)}{b-a}(x-a) + g(a).$$

We denote $\widetilde{f^\star}(x) := f^\star(x) - \ell(x)$ and $\widetilde{g}(x) := g(x) - \ell(x)$. Denote y_0 where $\widetilde{g}$ attains the minimum value on $[a,b]$. We prove that on a closed subinterval containing y_0, there exists an horizontal line $\tilde{\ell}$ such that the graph of $\widetilde{f^\star}$ is above and the graph of $\widetilde{g}$ is below. Notice that $\widetilde{g}(y_0)$ is negative since

$$\widetilde{g}(y_0) \le \widetilde{g}\left(\frac{a+b}{2}\right) < \frac{g(a)+g(b)}{2} - \ell(\frac{a+b}{2}) = 0.$$

Since $y_0 \in A$, it follows that $\widetilde{f^\star}(y_0) - \widetilde{g}(y_0) = f^\star(y_0) - g(y_0) =: C > 0$. For each $-\widetilde{g}(y_0) > \varepsilon > 0$ let

$$a_\varepsilon := \max\{a \le x \le y_0; \widetilde{g}(x) \ge \widetilde{g}(y_0) + \varepsilon\}$$

and

$$b_\varepsilon := \min\{y_0 \le x \le b; \widetilde{g}(x) \ge \widetilde{g}(y_0) + \varepsilon\}.$$

This definitions imply that for every $x \in [a_\varepsilon, b_\varepsilon]$, it holds $\widetilde{g}(x) \le \widetilde{g}(y_0) + \varepsilon$, with $\widetilde{g}(a_\varepsilon) = \widetilde{g}(b_\varepsilon) = \widetilde{g}(y_0) + \varepsilon$, and is strict in the interior. Now, we can find the desired horizontal line. Suppose that for every $\varepsilon > 0$ we can find $z_\varepsilon \in [a_\varepsilon, b_\varepsilon]$, such that $\widetilde{f^\star}(z_\varepsilon) < \widetilde{g}(y_0) + \varepsilon$. From the collection $\{z_\varepsilon\}_{\varepsilon>0}$, there is a subsequence that converges to some $z_0 \in [a,b]$ and $\widetilde{f^\star}(z_0) \le \widetilde{g}(y_0) \le \widetilde{g}(z_0) < \widetilde{f^\star}(z_0)$ which is a contradiction. Therefore, there exists an $\varepsilon > 0$ such that

$$\widetilde{f^\star}(x) \ge \widetilde{g}(y_0) + \varepsilon$$

for every $x \in [a_\varepsilon, b_\varepsilon]$. Now, observe that,

$$f^\star(x) \geq \ell(x) + \widetilde{g}(y_0) + \varepsilon \geq g(x), \tag{5.22}$$

for every $x \in [a_\varepsilon, b_\varepsilon]$ and the line $\ell(x) + \widetilde{g}(y_0) + \varepsilon$ connects the points $(a_\varepsilon, g(a_\varepsilon))$ and $(b_\varepsilon, g(b_\varepsilon))$ and the line is strictly above g in $(a_\varepsilon, b_\varepsilon)$.

Now define the sliced function

$$u(x) = \begin{cases} g(x), & \text{if } x \notin [a_\varepsilon, b_\varepsilon], \\ \frac{g(b_\varepsilon)-g(a_\varepsilon)}{b_\varepsilon - a_\varepsilon}(x - a_\varepsilon) + g(a_\varepsilon), & \text{if } x \in [a_\varepsilon, b_\varepsilon]. \end{cases}$$

The idea is to show that $u \in \mathcal{S}$, $u \succeq g$ and u is strictly bigger in $(a_\varepsilon, b_\varepsilon)$. The inequality (5.22) proves that $f^\star \succeq u \succeq f_0$ and $\mathrm{Lip}(u) \leq \mathrm{Lip}(g) \leq \mathrm{Lip}(f_0)$. It remains to prove that $||u'||_p \leq ||f_0'||_p$, but this follows from the fact that Young's inequality implies that

$$\begin{aligned}
||g'||_p^p &= \int_{\mathbb{R}\setminus[a_\varepsilon,b_\varepsilon]} |g'(x)|^p \, dx + \int_{a_\varepsilon}^{b_\varepsilon} |g'(x)|^p \, dx \\
&\geq \int_{\mathbb{R}\setminus[a_\varepsilon,b_\varepsilon]} |g'(x)|^p \, dx + (b_\varepsilon - a_\varepsilon) \left(\frac{1}{b_\varepsilon - a_\varepsilon} \int_{a_\varepsilon}^{b_\varepsilon} |g'(x)| \, dx \right)^p \\
&\geq \int_{\mathbb{R}\setminus[a_\varepsilon,b_\varepsilon]} |g'(x)|^p \, dx + (b_\varepsilon - a_\varepsilon) \left| \frac{1}{b_\varepsilon - a_\varepsilon} \int_{a_\varepsilon}^{b_\varepsilon} g'(x) \, dx \right|^p \\
&= \int_{\mathbb{R}\setminus[a_\varepsilon,b_\varepsilon]} |g'(x)|^p \, dx + (b_\varepsilon - a_\varepsilon) \left| \frac{g(b_\varepsilon) - g(a_\varepsilon)}{b_\varepsilon - a_\varepsilon} \right|^p \\
&= ||u'||_p^p.
\end{aligned}$$

This proves that u is an element in $\mathcal{S}$ and is strictly bigger than g, but this a contradiction since g is the maximal element in $\mathcal{S}$. In conclusion $g = f^\star$.

5.3.3 Proof of part (ii) - Theorem 5.1

The proof is as follows. We can assume $f_0 \in W^{1,1}(\mathbb{R})$ to be absolutely continuous. By Lemmas 5.3 and 5.4, $f^\star$ is continuous in $\mathbb{R}$ and subharmonic in the open set $A = \{x \in \mathbb{R}; f^\star(x) > f_0(x)\}$. As before we can regard A as a countable union of disjoint intervals $\cup_j I_j = \cup_j (\alpha_j, \beta_j)$. Since $f^\star$ is convex on every I_j, it is Lipschitz on every compact subinterval of A. This implies that $f^\star$ is differentiable almost everywhere on each I_j, lets denote this derivative on A as v.

Now, we prove that on each I_j, the variation of $f^\star$ is no greater than that of f_0. Since the function $f^\star$ is convex on I_j, it reaches a minimum, say $\gamma_j \in [\alpha_j, \beta_j]$. We do not discard unbounded intervals. If $\alpha_j = -\infty$, $f_0(\alpha_j) = 0$ since $f_0 \in W^{1,1}(\mathbb{R})$, and $f^\star(\alpha_j) \leq Mf(\alpha_j) = 0$, see Propositions 3.3 and 3.4. Similarly $f_0(\beta_j) = f^\star(\beta_j) = 0$, if $\beta_j = +\infty$. Hence, we have that $f^\star$ is monotone on the subintervals $[\alpha_j, \gamma_j]$ and $[\gamma_j, \beta_j]$ and integrating $|v|$ on I_j

$$\begin{aligned}
\int_{I_j} |v(x)|\, dx &= -\int_{\alpha_j}^{\gamma_j} v(x)\, dx + \int_{\gamma_j}^{\beta_j} v(x)\, dx \\
&= (f^\star(\alpha_j) - f^\star(\gamma_j)) + (f^\star(\beta_j) - f^\star(\gamma_j)) \\
&\leq (f_0(\alpha_j) - f_0(\gamma_j)) + (f_0(\beta_j) - f_0(\gamma_j)) \\
&\leq \int_{\alpha_j}^{\gamma_j} |f_0'(x)|\, dx + \int_{\gamma_j}^{\beta_j} |f_0'(x)|\, dx \\
&= \int_{I_j} |f_0'(x)|\, dx,
\end{aligned}$$

where we used the fact that $f^\star$ and f_0 coincides in the extremes of I_j and $f_0(\gamma_j) \leq f^\star(\gamma_j)$. This implies the inequality

$$\int_A |v(x)|\, dx \leq \int_A |f_0'(x)|\, dx. \tag{5.23}$$

It follows as in equation (3.6) that $f^\star$ is weakly differentiable and

$$(f^\star)' = \chi_{\mathbb{R}\setminus A} f_0' + \chi_A v. \tag{5.24}$$

The key point is that as in equation (3.7), it follows that for every $\phi \in C_c^\infty(\mathbb{R})$

$$\int_{I_j} f^\star(x)\phi'(x)\, dx = [f_0(\beta_j)\phi(\beta_j) - f_0(\alpha_j)\phi(\alpha_j)] - \int_{I_j} v(x)\phi(x)\, dx.$$

We can conclude that

$$\begin{aligned}
||(f^\star)'||_1 &= \int_{\mathbb{R}} |(f^\star)'(x)|\, dx \\
&= \int_{\mathbb{R}\setminus A} |(f_0)'(x)|\, dx + \int_A v(x)\, dx \\
&\leq \int_{\mathbb{R}\setminus A} |(f_0)'(x)|\, dx + \int_A |f_0'(x)|\, dx \\
&= ||(f_0)'||_1,
\end{aligned} \tag{5.25}$$

using in the first line the equation (5.24) and the inequality (5.23) in the last line.

5.3.4 Proof of part (iii) - Theorem 5.1

Remember that we want to prove that if f_0 is a function of bounded variation, then $f^\star$ has bounded variation and this is not larger than that of f_0. We can consider Df_0 as the weak derivative, being of bounded variation, it is a Radon measure with total variation $|Df_0|$. This satisfies $|Df_0| \leq \mathrm{Var}(f_0)$.

For $\varepsilon > 0$, we can consider one more time

$$f_\varepsilon(x) := f_0 * P_\varepsilon(x).$$

Because of the convolution, f_ε belongs to $C^\infty(\mathbb{R})$ and is also Lipschitz continuous. This also means that the total variation $|Df_\varepsilon|$ and the variation $\mathrm{Var}(f_\varepsilon)$ coincides. Again, we define

$$f_\varepsilon^\star(x) = \sup_{y>0} f_\varepsilon * P_y(x) = \sup_{y>\varepsilon} f_0 * P_y(x)$$

by the semigroup property of the Poisson kernel. From Lemma 5.4 we have that $f_\varepsilon^\star$ is subharmonic in the open set $A = \{x \in \mathbb{R}; f_\varepsilon^\star(x) > f_\varepsilon\}$. As usual we represent this set as the countable disjoint union $\cup_j I_j = \cup_j(\alpha_j, \beta_j)$. On each I_j, we have that $f_\varepsilon^\star$ is convex, so it is monotone on the pieces $[\alpha_j, \gamma_j]$ and $[\gamma_j, \beta_j]$, where $f_\varepsilon^\star$ attains its minimum in γ_j. We consider an arbitrary partition $\mathcal{P} = \{x_1, \ldots, x_N\}$ with $x_1 < x_2 < \cdots < x_N$, and we obtain a refinement $\mathcal{P}' = \{y_1, \ldots, y_M\}$ in a way that γ_j and the endpoints α_j, β_j are included. Hence, using the triangle inequality, we can compare the variation in both partitions and additionally, we observe that

$$\begin{aligned}
\mathrm{Var}_{\mathcal{P}}(f_\varepsilon^\star) &\leq \mathrm{Var}_{\mathcal{P}}(f_\varepsilon^\star) \\
&= \sum_{k=1}^{M-1} |f_\varepsilon^\star(y_{k+1}) - f_\varepsilon^\star(y_k)| \\
&\leq [f_\varepsilon^\star(\alpha_j) - f_\varepsilon^\star(\gamma_j)] + [f_\varepsilon^\star(\beta_j) - f_\varepsilon^\star(\gamma_j)] \\
&\leq [f_\varepsilon(\alpha_j) - f_\varepsilon(\gamma_j)] + [f_\varepsilon(\beta_j) - f_\varepsilon(\gamma_j)] \\
&\leq \mathrm{Var}(f_\varepsilon).
\end{aligned} \tag{5.26}$$

We have used the fact that $f_\varepsilon^\star$ and f_ε coincide in the endpoints of I_j and in the interior $f_\varepsilon < f_\varepsilon^\star$. The computations are similar in case I_j would be unbounded. In such case, $f_\varepsilon^\star$ is monotone and tends to zero at infinity. Since $Df_\varepsilon(x) = (Df_0) * P_y(x)$ and the Young's inequality we can deduce

$$\mathrm{Var}(f_\varepsilon) = |Df_\varepsilon| \leq |Df_0| \, ||P_\varepsilon||_1 = |Df_0| \leq \mathrm{Var}(f_0). \tag{5.27}$$

On one hand, $f_\varepsilon^\star \to f^\star$ pointwise as $\varepsilon \to 0$. On the other, the inequalities (5.26) and (5.27) implies $\mathrm{Var}_{\mathcal{P}}(f_\varepsilon^\star) \leq \mathrm{Var}(f_0)$ for any partition $\mathcal{P}$. Therefore,

$$\begin{aligned}\mathrm{Var}_{\mathcal{P}}(f^\star) &= \lim_{\varepsilon\to 0}\sum_{k=1}^{N-1}|f_\varepsilon^\star(x_{k+1}) - f_\varepsilon^\star(x_k)| \\ &= \lim_{\varepsilon\to 0}\mathrm{Var}_{\mathcal{P}}(f_\varepsilon^\star) \\ &\leq \mathrm{Var}(f_0).\end{aligned}$$

Being $\mathcal{P}$ an arbitrary partition, we have the desired result

$$\mathrm{Var}(f^\star) \leq \mathrm{Var}(f_0).$$

5.4 The regularity of heat flow maximal function: Proof of Theorem 5.2

The proof of this theorem is essentially the same as the proof of the Theorem 5.1. Remember that the heat flow maximal operator is given by the expression in (5.5). The first steps consist in proving an analogous to Lemma 5.3 and then obtain a result on the subharmonicity for f^* on set $\{x \in \mathbb{R}^d; f^*(x) > f_0(x)\}$. The rest of the proof follows in the same lines, we avoid to do this here since the details appear in [4].

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